TROPICAL EIGENWAVE AND INTERMEDIATE JACOBIANS

GRIGORY MIKHALKIN AND ILIA ZHARKOV

Abstract. Tropical manifolds are polyhedral complexes enhanced with certain kind of affine structure. This structure manifests itself through a particular cohomology class which we call the eigenwave of a tropical manifold. Other wave classes of similar type are responsible for deformations of the tropical structure.

If a tropical manifold is approximable by a 1-parametric family of complex manifolds then the eigenwave records the monodromy of the family around the tropical limit. With the help of tropical homology and the eigenwave we define tropical intermediate Jacobians which can be viewed as tropical limit of classical intermediate Jacobians.

1. Tropical spaces

Let us briefly recall the setup in which tropical manifolds are defined, see [Mik06] and [MR12] for details.

A tropical affine n-space $\mathbb{T}^n$ is the topological space $[-\infty,+\infty)^n$ (homeomorphic to the nth power of a half-open interval) enhanced with a preferred collection of functions $\mathcal{O}_{\text{pre}} = \{f\}$, $f : U \to \mathbb{T} = [-\infty,+\infty)$. Here $U \subset \mathbb{T}^n$ is an open set and f is a function that can be expressed as

$$f(x) = \max_{j\in A}(jx + a_j) \tag{1.1}$$

for a finite set $A \subset \mathbb{Z}^n$ and a collection of numbers $a_j \in \mathbb{T}$, such that the scalar product jx is well-defined as a number in $\mathbb{T}$ (i.e. is finite or $-\infty$) for any $x \in U$. The reason for calling it tropical is that in *tropical calculus* we replace addition by taking the maximum and multiplication by addition, so (1.1) becomes a tropical Laurent polynomial

$$f(x) = \text{“}\sum_{j\in A} a_j x^j\text{”},$$

where the quotation marks signify tropical arithmetic operations, and $x^j := x_1^{j_1}\dots x_n^{j_n}$ for $x = (x_1,\dots,x_n)$ and $j = (j_1,\dots,j_n)$.

The collection of functions $\mathcal{O}_{\text{pre}}$ is a presheaf which gives rise to a sheaf $\mathcal{O}$ of regular functions on $\mathbb{T}^n$. We have $f \in \mathcal{O}(U)$ if f is *locally* given by a tropical Laurent polynomial. Such functions are called *regular functions* and $\mathcal{O}$ (which we

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will also denote $\mathcal{O}_{\mathbb{T}^n}$ indicating the space where it is defined to avoid ambiguity) is called the *structure sheaf*.

Note that to write down tropical Laurent polynomials (1.1) inside $\mathbb{R}^n$ all we need is the *affine structure* on $\mathbb{R}^n$, where we distinguish affine functions $\mathbb{R}^n \to \mathbb{R}$ as long as their linear parts are defined over $\mathbb{Z}$. Such functions correspond exactly to tropical Laurent monomials.

Thus the tropical structure on $\mathbb{T}^n$ can be thought of as an extension of the integral affine structure in $\mathbb{R}^n$ where the overlapping maps are compositions of linear transformations in $\mathbb{R}^n$ defined over $\mathbb{Z}$ with arbitrary translations in $\mathbb{R}^n$. Similarly, tropical structures on general tropical manifolds can be thought of as extensions of such integer affine structure to more general n-dimensional polyhedral complexes.

An n-dimensional polyhedral complex $X = \bigcup D$ is the union of a finite collection of convex n-dimensional polyhedral domains $D \subset \mathbb{T}^N$ with integer slopes. This means that D is the intersection $\bigcap\limits_k H_k$ of a finite collection of half-spaces H_k of the form

$$H_k = \{x \in \mathbb{T}^n \mid jx \leq a\} \subset \mathbb{T}^n \supset \mathbb{R}^n \tag{1.2}$$

for some $j \in \mathbb{Z}^n$ and $a \in \mathbb{R}$ and is n-dimensional as the topological space. The boundary ∂H_k is given by the equation $jx = a$.

Each such n-dimensional polyhedral domain D is called a *facet* of X. A *mobile face* E of D is the intersection of D with the boundaries of some of its defining half-spaces given by (1.2). We require that the intersection of any collection of facets of X is their common mobile face. The adjective *mobile* stands here to distinguish such faces among more general faces of X (which we will define later) that are allowed to have support in $\mathbb{T}^N \setminus \mathbb{R}^N$, i.e. be disjoint from $\mathbb{R}^N$. The domains D in the union X are called *facets* of X. We require that facets intersect along their common mobile faces E in the polyhedral complex X.

This construction of the polyhedral complex may be considered abstractly or as embedded in the Euclidean space $\mathbb{R}^N$ for some N or in its partial compactification $\mathbb{T}^N$. Note that *a priori* we consider X as an abstract polyhedral complex, so it does not come with a fixed embedding. Nevertheless, locally the tropical structure on X always comes from some embedding to $\mathbb{T}^N$.

Namely, our X is covered by open sets U_α equipped with open embeddings $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$. Here each $Y_\alpha \subset \mathbb{T}^N$ is an embedded n-dimensional polyhedral complex in $\mathbb{T}^N$ subject to the following *balancing condition* for each $(n-1)$-face E of Y_α.

Condition (Balancing)**.** Let $E \subset \mathbb{T}^{N_\alpha}$ be an $(n-1)$-dimensional mobile face of Y_α and $D_1, \dots, D_l \subset \mathbb{T}^{N_\alpha}$ be the facets adjacent to E. Take the quotient of $\mathbb{R}^{N_\alpha}$ by the

linear subspace parallel to E. The balancing condition asserts that

$$\sum_{k=1}^{l} \epsilon_k = 0, \tag{1.3}$$

where the ϵ_k are the outward primitive integer vectors parallel to the D_k in this quotient.

The polyhedral complex Y_α inherits its structure sheaf $\mathcal{O}_{Y_\alpha}$ as the restriction of $\mathcal{O}_{\mathbb{T}^{N_\alpha}}$. In turn, its pull back induces a sheaf on U_α. Two charts $\phi_\alpha : U_\alpha \to Y_\alpha \subset \mathbb{T}^{N_\alpha}$ and $\phi_\beta : U_\beta \to Y_\beta \subset \mathbb{T}^{N_\beta}$ are called compatible if the sheaves induced by ϕ_α and ϕ_β on $U_\alpha \cap U_\beta$ coincide.

Definition 1.1. A tropical space is a polyhedral complex (treated as a topological space) enhanced with a cover of compatible charts $\phi_\alpha : U_\alpha \to \mathbb{T}^{N_\alpha}$ that are open embeddings to balanced polyhedral complexes $Y_\alpha \subset \mathbb{T}^{\mathbb{N}_\alpha}$.

As we treat X only as a topological space, its subdivision into faces is not fixed. We are free to choose any other subdivision compatible with the charts ϕ_α. The charts induce a sheaf $\mathcal{O}_X$ on X which we call the structure sheaf of X. Alternatively we may define a tropical space as a pair $(X, \mathcal{O}_X)$ locally given by balanced polyhedral complexes Y_α.

In addition we will impose the following *finite type* condition on X.

Condition (Finite type)**.** If $\{x_j \in U_\alpha\}_{j=1}^\infty$ is a sequence such that $\phi_\alpha(x_j)$ converges to a point $y \in \mathbb{T}^{N_\alpha}$ then either the sequence $\{x_j\}$ itself converges to a point $x \in X$ or there exists a coordinate in $\mathbb{T}^{N_\alpha}$ taking the $(-\infty)$-value on y and finite values on some $\phi_\alpha(x_j)$.

Remark 1.2. Recall that the polyhedral complex X is already required to be finite. Thus our finite type condition is equivalent to the finite type condition from [MR12]. On the other hand here we use a more restrictive notion of tropical space than one in [MR12] as here we do not allow facets to be weighted.

Note that the dimensions N_α and N_β of the ambient spaces may differ. Furthermore, given a point $x \in X$ some coordinates of $\phi_\alpha(x)$ may take infinite values. So sometimes the point $\phi_\alpha(x)$ is not contained in $\mathbb{R}^{N_\alpha}$. It is convenient to stratify the space $\mathbb{T}^N$ by

$$\mathbb{T}_I^\circ := \{y \in \mathbb{T}^N \ : \ y_i = -\infty, i \in I \ \text{ and } \ y_i > -\infty, i \notin I\},$$

where $I \subset \{1, \ldots, n\}$. Each T_I° is isomorphic to $\mathbb{R}^{N-|I|}$ and we set $\mathbb{T}_I$ to be its closure in $\mathbb{T}^N$.

Let us look again at the faces of the polyhedral complexes Y_α and X. An n-dimensional polyhedral complex is composed of its facets that are convex polyhedral domains $D \subset \mathbb{T}^N$. We may think of each D as the topological closure in $\mathbb{T}^N$ of a

convex n-dimensional polyhedral domain in $\mathbb{R}^N$ and of its faces E as the closures in $\mathbb{T}^N$ of the corresponding faces of that domain in $\mathbb{R}^N$.

It is convenient to extend the notion of a face to include the intersections $E \cap \mathbb{T}_I$ for $I \subset \{1, \dots, n\}$.

Definition 1.3. We say the face $E_I := E \cap \mathbb{T}_I$ has *sedentarity* $s = |I|$ or *refined sedentarity* I. Clearly, the *mobile* faces (defined previously) are the faces of sedentarity 0. If $E_i := E \cap \mathbb{T}_i \neq \emptyset$ we say that the coordinate y_i defines the *divisorial* direction in E, see Figure 1. The primitive integral vector along y_i pointing towards $-\infty$ is called a *divisorial vector*.

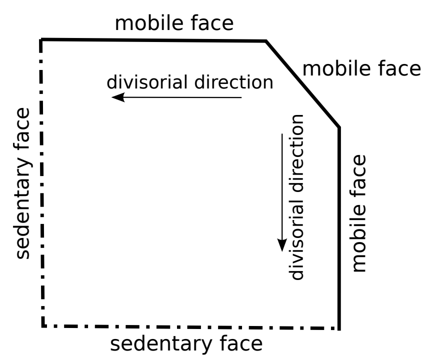


FIGURE 1. Mobile and sedentary faces of a polyhedral domain in $\mathbb{T}^N$.

We will use Δ to denote these more general faces of X. The collection of the faces of X forms a partially ordered set by inclusion. Indeed, facets are the maximal faces. Recall that all facets in X are required to intersect along their common subfaces. This implies that the same property also holds for all faces, including faces of positive sedentarity. Thus the relative interiors of the faces define a stratification of X.

As an example let us look at the tropical affine space $\mathbb{T}^N$ as a polyhedral complex in itself. Its faces are the $\mathbb{T}_I$. Then the sedentarity of a point $x \in \mathbb{T}^N$ agrees with the sedentarity of the face $\mathbb{T}_I$ containing x in its relative interior. Note that any translation in $\mathbb{R}^N$ extends to a continuous homeomorphism of $\mathbb{T}^N$. We call such extensions *translations in* $\mathbb{T}^N$, they form a group isomorphic to $\mathbb{R}^N$.

Clearly, the dimension of the stabilizer of a point x coincides with its sedentarity. In this sense the higher the sedentarity of the point the less mobile it is. This explains the term sedentarity. The same is applicable to the faces of X.

Definition 1.4. A face Δ of a polyhedral subdivision of X is called *infinite* if either it is not compact or it contains a subface of sedentarity higher than Δ. Otherwise Δ is called finite (even if the sedentarity of Δ itself is positive).

Proposition 1.5. *If X is compact then each face $\Delta \subset X$ has a unique subface of maximal sedentarity. Moreover, this maximal sedentary face is finite.*

Proof. If Δ is infinite and X is compact then it has a subface of higher sedentarity. But if the closure of Δ contains two points x, y of different sedentarity then by convexity it also contains a point z whose refined sedentarity contains the refined sedentarities of x and y. (Recall that all our facets are are compact and contained in a single chart in some $\mathbb{T}^N$.) $\square$

Let us assume now that X is compact. We conclude this section by giving a construction of the first baricentric subdivision of X. For a finite cell we take an arbitrary point in its interior for its baricenter. For an infinite cell we take for its

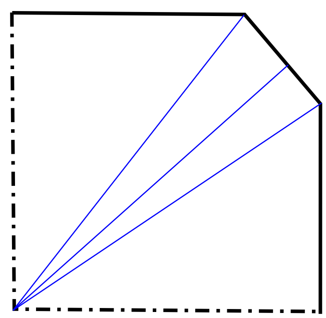


FIGURE 2. Baricentric subdivision of an infinite cell. The dotted faces have higher sedentarity.

baricenter the baricenter of its unique most sedentary (necessarily finite, cf. Proposition 1.5) subface (see Figure 2). That is, we first choose baricenters of maximal sedentary faces and then name them also as baricenters of any adjacent faces of lower sedentarity. The subdivision of each face of X into simplices is constructed as usual by the flags of its subfaces of *minimal* sedentarity.

2. Homology groups

2.1. Singular tropical homology.

Definition 2.1. Let $x \in X$ be a point in a tropical space. Choose a sufficiently small open set $U \ni x$ and an embedding $\phi : U \to Y \subset \mathbb{T}^N$. We define $\mathcal{F}(x)$ as the free abelian group generated by the integer vectors in $\mathbb{T}^\circ_{I(\phi(x))}$ parallel to $\phi(U) \cap \mathbb{T}^\circ_{I(\phi(x))}$. This means that $\mathcal{F}(x)$ is generated by the integer vectors parallel to the faces of the polyhedral complex X adjacent to the point x.

The group $\mathcal{F}_k(x)$ is defined as the subgroup of the kth exterior power $\Lambda^k(\mathcal{F}(x))$ generated by the exterior products $v_1 \wedge \cdots \wedge v_k$ with $v_1, \ldots, v_k \in \mathcal{F}(y)$ for a point y in a relative interior of a facet of X adjacent to x. It is important that all k elements v_j come from a single adjacent facet. The group $\mathcal{F}^k(x)$ is defined as $\mathrm{Hom}(\mathcal{F}_k(x), \mathbb{Z})$.

It is easy to see (cf. [MR12]) that for a sufficiently small open set $U \ni x \in X$ (namely, if U is contained in the open star of x) the groups $\mathcal{F}_k(x)$ do not depend

on the choice of U or ϕ as they can be expressed in intrinsic terms of the sheaf $\mathcal{O}_X$ through invertible regular functions in a neighborhood of x.

On the other hand, the groups $\mathcal{F}_k(x)$ and $\mathcal{F}_k(y)$ are canonically identified by translation if x and y belong to the relative interior of the same face Δ of X. Thus we can use the notation $\mathcal{F}_k(\Delta)$. Furthermore, if for two points x, y we have $\Delta_x \succ \Delta_y$ then there are natural homomorphisms

$$\iota : \mathcal{F}_k(x) \to \mathcal{F}_k(y). \tag{2.1}$$

To define the map (2.1) we take a chart ϕ with $U \ni y$. If $I(\phi(y)) = I(\phi(x))$ then any face adjacent to $\phi(x)$ is contained in some face adjacent to $\phi(y)$ and the inclusion induces the required map. If $I(\phi(y)) \neq I(\phi(x))$ (note that we must have $I(\phi(y)) \supset I(\phi(x))$) then the required map is given by projection along the divisorial directions indexed by $I(\phi(y)) \smallsetminus I(\phi(x))$.

If there are three points x, y, z lying in the faces with incidence $\Delta_x \succ \Delta_y \succ \Delta_z$ then the three corresponding maps (2.1) form a commutative diagram. In other words, if we consider the set of faces of X as a category (under inclusions) then the $\mathcal{F}_k$ forms a contravariant functor from faces of X to abelian groups (cf. Proposition 2.14).

Note that $\mathcal{F}_k$ does not depend on the choice of a subdivision of X into faces, i.e. on a presentation of X as a polyhedral complex, because $\mathcal{F}_k$ is constant on relative interiors of faces for any subdivision of X (compatible with the charts ϕ_α). It is then convenient to consider the coarsest stratification of X compatible with all such subdivisions.

Definition 2.2. We say that two points $x, x' \in X$ are *combinatorially equivalent* if there exists a finite sequence of points $x = x_0, x_1, \ldots, x_l = x'$, the charts $\phi_j : U_j \to Y_j \subset \mathbb{T}^{N_j}$, and the subdivisions of the Y_j into polyhedral faces, such that for each $j = 1, \ldots, l$, the pair $x_{j-1}, x_j \in U_j$ and their ϕ_j-images belong to the same face of Y_j (in particular, x_{j-1}, x_j have the same sedentarity). *A combinatorial stratum* of the tropical space X is a class of combinatorial equivalence.

Example 2.3. Consider the circle E_l of length l, the so-called *tropical elliptic curve.* We can present E_l as the quotient of $\mathbb{R}$ by the group generated by translation by l units. Clearly, X_l is a tropical space and we can present it as a polyhedral complex by choosing e.g. three distinct points so that they split E_l into three facets. This subdivision is not unique as we can move these points around or consider a subdivision into a larger number of facets. The combinatorial stratification for E_l is trivial: it consists of a single stratum E_l.

Clearly we have $\mathcal{F}_k(x) = \mathcal{F}_k(x')$ for combinatorially equivalent points x, x'. In other words, $\mathcal{F}_k$ associates an abelian group to each combinatorial stratum of X.

We may interpret our data as a system of coefficients suitable to define homology groups on X. It is not locally constant as it jumps when we pass from a face of X to its subface. But this does not pose a problem as long as we consider singular

chains compatible with the combinatorial stratification of X. Namely, we consider the finite formal sums

$$\sum \beta_\sigma \sigma,$$

where each singular simplex $\sigma : \Delta^q \to X$ is such that for each relatively open face Δ' of Δ^q the image $\sigma(\Delta')$ is contained in a single combinatorial stratum of X. Slightly abusing the notations we'll identify the source and the image of σ with the singular simplex σ itself and say that $\tau = \sigma|_{\Delta'}$ is a face of σ. Here $\beta_\sigma \in \mathcal{F}_k(\Delta_\sigma)$. These chains form a complex $C_\bullet(X; \mathcal{F}_k)$ with the differential ∂ given by the standard singular differential followed by the maps 2.1. We call such compatible singular chains with coefficients in $\mathcal{F}_k$ *tropical chains*. The groups

$$H_{p,q}(X) = H_q(C_\bullet(X; \mathcal{F}_p), \partial)$$

are called the *tropical homology* groups.

These homology groups is a version of singular homology groups of a topological space X (after imposing the condition of compatibility of singular chains with the faces of X). There are other equivalent ways for constructing tropical homology groups: simplicial, cellular and Čech.

2.2. **Cellular and simplicial tropical homology.** We assume X is compact throughout this subsection. The main advantage of dealing with cellular and simplicial chain groups is that they are finitely generated. This will give an effective way to calculate the tropical homology.

We fix a subdivision of X into convex polyhedral domains (which is by no means unique). Sometimes the combinatorial stratification provides a natural cellular structure on X, but this is not always the case, cf. Example 2.3. Recall that we call the faces (including the sedentary faces) of these polyhedral domains the faces Δ of X. Note that such subdivision is always compatible with the combinatorial stratification in the sense that the relative interior of each Δ is contained in a single combinatorial stratum of X.

Once such a subdivision is fixed we define the cellular chain complex

$$C_q^{cell}(X; \mathcal{F}_p) = \oplus \mathcal{F}_p(\Delta) = \oplus H_q(\Delta, \partial\Delta; \mathcal{F}_p(\Delta)).$$

Here the direct sum is taken over all q-dimensional faces Δ of the subdivision. The homology $H_q(\Delta, \partial\Delta; \mathcal{F}_p(\Delta))$ of the pair with constant coefficients equals $\mathcal{F}_p(\Delta)$ since each q-dimensional face Δ in X is topologically a closed q-disk (recall that X is compact).

Our next step is to define the boundary homomorphism $\partial : C_q^{cell}(X; \mathcal{F}_p) \to C_{q-1}^{cell}(X; \mathcal{F}_p)$. The ∂ is the composition of the maps

$$H_q(\Delta, \partial\Delta; \mathcal{F}_p(\Delta)) \to H_{q-1}(\partial\Delta; \mathcal{F}_p(\Delta)) \to H_{q-1}(\partial\Delta, \partial\Delta \cap \mathrm{Sk}_{q-2}(X); \mathcal{F}_p(\Delta)), \tag{2.2}$$

the isomorphism

$$H_{q-1}(\partial\Delta, \partial\Delta \cap \mathrm{Sk}_{q-2}(X); \mathcal{F}_p(\Delta)) \to \oplus H_{q-1}(\Delta', \partial\Delta'; \mathcal{F}_p(\Delta)), \tag{2.3}$$

where the direct sum is taken over all $(q-1)$-dimensional subfaces $\Delta' \prec \Delta$, and

$$\oplus H_{q-1}(\Delta', \partial\Delta'; \mathcal{F}_p(\Delta)) \to \oplus H_{q-1}(\Delta', \partial\Delta'; \mathcal{F}_p(\Delta')). \tag{2.4}$$

In (2.2) the first homomorphism is the boundary homomorphism of the pair $(\Delta, \partial\Delta)$ and the second one is induced by the inclusion of the pairs $(\Delta, \emptyset) \subset (\Delta, \partial\Delta)$. The isomorphism (2.3) comes from the excision as the quotient space $\partial\Delta/(\partial\Delta \cap \mathrm{Sk}_{q-2}(X))$ is homeomorphic to a bouquet of $(q-1)$-dimensional spheres, one sphere for each $(q-1)$-dimensional subface $\Delta' \prec \Delta$. Finally, the homomorphism (2.4) is induced by (2.1).

The homology groups of the cellular chain complex $(C^{cell}_\bullet(X; \mathcal{F}_p), \partial)$ are called the cellular tropical homology groups $H^{cell}_\bullet(X; \mathcal{F}_p)$. We have the following identification.

Proposition 2.4. *The cellular tropical homology groups $H^{cell}_\bullet(X; \mathcal{F}_p)$ are canonically isomorphic to the (singular) tropical homology groups $H_\bullet(X; \mathcal{F}_p)$.*

Proof. As in algebraic topology with constant coefficients to prove this isomorphism we need to use cellular homotopy. Let us recall that by the cellular homotopy argument the inclusion $\mathrm{Sk}_q(X) \to X$ induces an epimorphism

$$H_j(\mathrm{Sk}_q(X); \mathcal{F}_p) \to H_j(X; \mathcal{F}_p) \tag{2.5}$$

for $j \le q$ (which is also an isomorphism for $j < q$. Note that even though $\mathcal{F}_p$ is not a constant coefficient system, all cellular homotopy takes place within a single cell, so the classical argument also holds here.

Consider the homomorphism (in singular homology groups) induced by the inclusion of pairs $(\mathrm{Sk}_q(X), \emptyset) \subset (\mathrm{Sk}(X), \mathrm{Sk}_{q-1}(X))$

$$H_q(X; \mathcal{F}_p) \to H_q(\mathrm{Sk}_q(X), \mathrm{Sk}_{q-1}(X); \mathcal{F}_p) = C^{cell}_q(X; \mathcal{F}_p).$$

Its image consists of cycles by the construction of the boundary map in the short exact sequence of the pair and thus it gives us a homomorphism

$$H_q(\mathrm{Sk}_q(X); \mathcal{F}_p) \to H^{cell}_q(X; \mathcal{F}_p). \tag{2.6}$$

Note that by cellular homotopy the kernel of (2.6) coincides with the kernel of (2.5) for $j = q$. To see surjectivity of (2.6) we consider an element $c \in H^{cell}_q(X; \mathcal{F}_p)$. Subdividing the faces of X into simplices if needed we may represent c by a singular chain in $C_\bullet(\mathrm{Sk}_q(X); \mathcal{F}_p)$, whose boundary ∂c is null-homologous in

$$C^{cell}_{q-1}(\mathrm{Sk}_{q-1}(X), \mathrm{Sk}_{q-2}(X); \mathcal{F}_p).$$

But $H_{q-1}(\mathrm{Sk}_{q-2}(X); \mathcal{F}_p) = 0$ by the dimensional reason and thus ∂c must also vanish in $H^{cell}_{q-1}(\mathrm{Sk}_{q-1}(X); \mathcal{F}_p)$. Thus we may correct c (by adding to it a singular chain in $\mathrm{Sk}_{q-1}(X)$ whose boundary coincides with ∂c) to make it a cycle in $C_\bullet(X; \mathcal{F}_p)$. $\square$

We continue to assume that X is compact. There is a simplicial variant of the tropical homology arising from the first baricentric simplicial chains on X. (The baricentric subdivision of X was described at the end of the previous section.) Then

we can consider the baricentric simplicial chain complex with coefficients in $\mathcal{F}_p$ as a subcomplex $C^{bar}_\bullet(X;\mathcal{F}_p)$ of $C_\bullet(X;\mathcal{F}_p)$.

Note that the cellular chain complex $C^{cell}_\bullet(X;\mathcal{F}_p)$ can be viewed as a subcomplex of $C^{bar}_\bullet(X;\mathcal{F}_p)$, where all coefficients on simplices of the same cell are taken equal. Applying the standard chain homotopy arguments for constant coefficients one can show that this inclusion

$$C^{cell}_\bullet(X;\mathcal{F}_p) \hookrightarrow C^{bar}_\bullet(X;\mathcal{F}_p)$$

is again a quasi-isomorphism. This allows us to identify both baricentric simplicial and cellular homology with the tropical homology.

Remark 2.5. In Section 6 we will show that there is a fairly small subcomplex of $C^{bar}_\bullet(X;\mathcal{F}_p)$, called konstruktor, which suffices to calculate the homology groups $H_{p,q}(X)$ in the smooth projective realizable case.

Remark 2.6. In [IKMZ12] it is shown that in the case when X is a smooth projective tropical manifold that comes as the limit of a complex 1-parametric family the groups $H_{p,q}(X)$ can be obtained from the limiting mixed Hodge structure of the approximating family. In particular, we have the equality

$$h^{p,q}(X_t) = \operatorname{rk} H_{p,q}(X),$$

for the Hodge numbers $h^{p,q}(X_t)$ of a generic fiber X_t from the approximating family.

2.3. **Special tropical cycles.**

Definition 2.7. A cycle

$$\gamma = \sum \beta_\sigma \sigma \in C_p(X;\mathcal{F}_p)$$

is called *special* (or, *straight* cf. [MR12], [Sh12]) if

- the restriction of each singular simplex σ to its relative interior $\sigma^\circ \subset \sigma$ is an open embedding to a p-dimensional integral affine subspace L of the affine space $\langle\Delta_\sigma\rangle_{\mathbb{R}}$ spanned by Δ. Let $\Lambda_L := L \cap \langle\Delta_\sigma\rangle_{\mathbb{Z}} \subset \mathcal{F}_1(\Delta_\sigma)$ be the corresponding lattice.
- Each coefficient $\beta_\sigma \in \mathcal{F}_p(\Delta_\sigma)$ is an integer multiple of the integral volume element Vol_L of the lattice Λ_L.

Note that the volume element Vol_L is determined by the orientation of σ and vice versa. Thus the ratio between β_σ and Vol_L is a well-defined integer. We call it the *weight* of σ in γ and denote by $w(\sigma) \in \mathbb{Z}$.

It is convenient to present tropical cycles in the form independent of presentation by singular chains. Let us consider the support $Z = |\gamma| \subset X$ (i.e. the image of all simplices σ from γ). It is contained in the union of finitely many p-dimensional affine polyhedral subsets in different faces Δ of X and can be thought of as a generalized polyhedral complex.

Namely, we define a coarse facet A_L of Z as the union of all simplices $\sigma \subset \Delta$ of γ contained in a given p-dimensional affine subspace L. Each A_L is a polyhedral (possibly non-convex) domain in L. Indeed, as $\partial\gamma = 0$, the boundary of A_L in L, must be contained in an intersection $L \cap L'$ for some other p-dimensional affine subspace L' (perhaps from an adjacent face Δ' of X). Thus the relative boundaries of A_L can be decomposed into a union of $(p-1)$-dimensional polyhedral domains (again not necessarily convex) contained in $(p-1)$-dimensional affine subspaces with integer slopes.

We define the *weight* $w(x)$ of a generic point $x \in Z$ to be the sum of the weights of all p-simplices σ whose interiors contain x. Here we call x generic if it does not sit on the boundary of a simplex from γ and all these simplices are contained in a single p-dimensional space $L \subset \Delta$. The set of all generic points is an open and dense set in Z. Some points which are non-generic in this sense might still be generic for another cycle with the same support Z while some stay non-generic for any such cycles.

We say that a point $x \in Z$ is a *junction point* if any neighborhood $U \ni x$ contains a pair of generic points not contained in the same p-dimensional affine space $L \subset \Delta$. Thus if x is not a junction point its small neighborhood is contained in some p-dimensional affine space L. The *junction locus* $\mathrm{Sk}^{p-1}(Z) \subset Z$ is the set of all junction points. Clearly, the junction locus is supported on pairwise intersections of the coarse facets A_L of Z and thus can be presented as a union of $(p-1)$-dimensional polyhedral domains (again not necessarily convex) contained in $(p-1)$-dimensional integral affine subspaces. Note that since γ is a cycle all generic points in a small neighborhood of a non-junction point must have the same weight and thus we can extend the weight to a a locally constant function

$$w : Z \smallsetminus \mathrm{Sk}^{p-1}(Z) \to \mathbb{Z}.$$

The topological closures of the connected components of $Z \smallsetminus \mathrm{Sk}^{p-1}(Z)$ define facets of a polyhedral subdivision which refined the subdivision into A_L. These new facets are called *elementary facets* of Z. Each elementary facet is prescribed a single weight (equal to the weight of any generic point in it).

Definition 2.8. We say that a finite union $Y = \bigcup F$ of closed subsets $F \subset X$ (called facets of Y) is a *p-dimensional polyhedral pseudocomplex* $Y \subset X$ if the following conditions hold.

- Each facet F is a p-dimensional polyhedral domain (i.e. can be presented as the union of a finite number of convex n-dimensional polyhedral domains) contained in a p-dimensional integral affine subspace L of a face Δ of the tropical manifold X. The interior of F in L is non-empty and connected.
- Its boundary ∂F (as a subset of L) is contained in a finite union of $(p-1)$-dimensional affine subspaces $K_j \subset L$. A point $x \in \partial F$ is called a generic boundary point of F if it is contained in the relative interior of facet of F

(corresponding to some K_j). We say that an integer vector $v_x \in \mathcal{F}_1(\Delta)$ is an *outward primitive vector to F at $x \in \partial F$* if v_x points from x toward F and together with a basis of Λ_{K_j} it generates the lattice Λ_L.

The pseudocomplex Y is called *weighted* if each facet $F \subset Y$ is prescribed an integer number $w(F)$, the *weight* of F.

The junction locus $\mathrm{Sk}^{p-1}(Y) \subset Y$ is defined as the union of ∂F over all facets of Y. A point $x \in \mathrm{Sk}^{p-1}(Y)$ is called a *generic junction point* if it is a generic boundary point of F for every facet F such that $x \in \partial F$ and, in addition, there exists a $(p-1)$-dimensional affine space $K(x)$ in a face Δ of X such that a small neighborhood of x in $\mathrm{Sk}^{p-1}(Y)$ is contained in K. Note that such $K(x)$ is unique.

Definition 2.9. A p-dimensional weighted polyhedral pseudocomplex $Y \subset X$ is called *balanced* if for every generic junction point $x \in \mathrm{Sk}^{p-1}(Y)$ the vector

$$\sum_{F_j} w(F_j) v_x(F_j) \in \mathcal{F}_1(x)$$

is parallel to $K(x)$. Here the sum is taken over all facets F_j of Z such that $x \in \partial F_j$ and $v_x(F_j)$ is the outward primitive vector to F_j at $x \in \partial F_j$.

Remark 2.10. Note that if x has zero sedentarity, and $\phi : U \to \mathbb{R}^N$ is a chart with $U \ni x$, and $\phi(U \cap Y) \subset \mathbb{R}^N$ is a p-dimensional polyhedral complex then the balancing condition of Definition 2.9 coincides with the condition (1.3) for the $(p-1)$-face containing $\phi(x)$.

Proposition 2.11. *The support $|\gamma|$ of a special tropical (p,p)-cycle γ is a balanced p-dimensional weighted polyhedral pseudocomplex. Conversely, any p-dimensional balanced weighted polyhedral pseudocomplex Y can be presented as the support of a special tropical (p,p)-cycle.*

Proof. The balancing condition for $|\gamma|$ is equivalent to the condition $\partial\gamma = 0$. To find a special tropical (p,p)-cycle for a balanced weighted polyhedral pseudocomplex Y we take a compatible triangulation of the facets of Y. Then we take the volume element times the weight for the coefficients of each resulting simplex. Note that the orientation of each simplex of the triangulation enters the resulting tropical chain twice: once as the orientation of the simplex itself and once for through the orientation of the volume element. Thus the result does not depend on the choice of orientation of the simplices. ☐

Definition 2.12. Homology classes of special tropical cycles are also called special, or straight or algebraic tropical (p,p)-classes. They form a subgroup

$$H^{special}_{p,p}(X) \subset H_{p,p}(X).$$

2.4. **Tropical cohomology groups.** Finally we define tropical *cochains* $C^\bullet(X;\mathcal{F}^p)$ to be certain linear functionals on face-compatible singular chains with values in $\bigoplus_{\Delta \text{ faces of } X}\mathcal{F}^p(\Delta)$. Namely, if a simplex σ lies in a face Δ_σ then we require the value of the cochain to lie in $\mathcal{F}^p(\Delta_\sigma)$. Then one can define the differential as the usual coboundary followed by the maps dual to (2.1)

$$\delta\alpha(\sigma)=\alpha(\partial\sigma)\in\bigoplus_{\tau\subset\sigma}\mathcal{F}^p(\Delta_\tau)\to\mathcal{F}^p(\Delta_\sigma).$$

We can define the *tropical cohomology* groups

$$H^{p,q}(X)=H^q(C^\bullet(X;\mathcal{F}^p),\delta).$$

2.5. **Sheaf/cosheaf (co)homology.** To make connections with other homology theories we use the coefficient systems $\mathcal{F}_p$ to define a constructible cosheaf with respect to the stratification given by the face structure on X. With a slight abuse of notations we denote this cosheaf also by $\mathcal{F}_p$. A cosheaf is a suitable notion to take homology, just like sheaf – for cohomology.

Given an open set $U\subset X$ we consider the poset formed by the connected components of intersections of the faces of X with U. The order is given by adjacency. This poset can be represented by a quiver (oriented graph) $\Gamma(U)$. Each vertex $v\in\Gamma(U)$ corresponds to a connected component of the intersection $U\cap E$ of the open set U and a face E of X. A face of X can have multiple representatives in $\Gamma(U)$, see Figure 3. To each vertex v we associate the coefficient group $\mathcal{F}_p(v)=\mathcal{F}_p(E)$. To an arrow from v to w we associate the relevant homomorphism $i_{vw}:\mathcal{F}_p(v)\to\mathcal{F}_p(w)$ from (2.1). The groups $\mathcal{F}_p(v)$ with maps i_{vw} thus form a representation of the quiver $\Gamma(U)$.

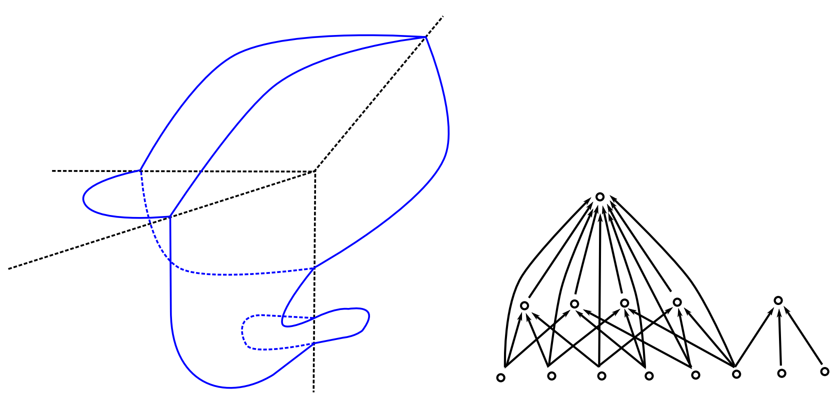


FIGURE 3. An open set in a polyhedral complex and the corresponding quiver. Here $\mathcal{F}_1(U)\cong\mathbb{Z}^4$.

Definition 2.13. $\mathcal{F}_p(U)$ is the quotient of the direct sum $\bigoplus_{v\in\Gamma(U)}\mathcal{F}_p(v)$ by the subgroup generated by the elements $a-i_{vw}(a)$ for all pairs of connected vertices (v,w), and all $a\in\mathcal{F}_p(v)$.

Note that an inclusion $U \subset V \subset X$ induces a morphism between the corresponding quivers $\Gamma(U) \to \Gamma(V)$ with isomorphisms at the corresponding vertices. This map clearly preserves the equivalence relation, and hence descends to the map $\mathcal{F}_p(U) \to \mathcal{F}_p(V)$. Thus, we get a covariant functor from the open sets $U \subset X$ (with morphism given by inclusions) to free abelian groups $U \mapsto \mathcal{F}_p(U)$. It is easy to check that all sequences

$$\bigoplus_{\alpha,\beta} \mathcal{F}_p(U_\alpha \cap U_\beta) \to \bigoplus \mathcal{F}_p(U_\alpha) \to \mathcal{F}_p(U) \to 0, \tag{2.7}$$

where $U = \bigcup U_\alpha$, are exact. Thus the functor $U \mapsto \mathcal{F}^p(U)$ is a cosheaf (cf., e.g. [Br97]).

To define the sheaf $\mathcal{F}^p$ we need a contravariant functor $U \mapsto \mathcal{F}^p(U)$. Let $\Gamma(U)$ to be the directed graph as before with all arrows reversed. We set $\mathcal{F}^p(U)$ to be the subgroups of $\bigoplus_{v\in\Gamma(U)} \mathcal{F}^p(v)$, where the collections of elements $\{a_v \in \mathcal{F}^p(v)\}$ are compatible with all the morphisms dual to (2.1). Note that these collections are precisely the ones annihilated by the elements $a - i_{vw}(a)$ from the Definition 2.13, and thus $F^p(U) = \mathrm{Hom}(\mathcal{F}_p(U), \mathbb{Z})$. Dualizing the exact sequences (2.7) we see that the functor $U \mapsto \mathcal{F}^p(U)$ is a (constructible) sheaf on X.

The faces of X also form a category. Its objects are the faces themselves. There is a unique morphism from E to E' if E is a subface of E', and no morphisms otherwise. Our reasoning above can be formulated as the following proposition.

Proposition 2.14. *Every covariant functor $\mathcal{F}$ from the faces of X yields a constructible sheaf while every contravariant functor yields a constructible cosheaf where $U \mapsto \mathcal{F}(U)$ is given by Definition 2.13.*

Finally we can use the sheaf-theoretic or Čech homology and cohomology for cosheaves $\mathcal{F}_p$ and sheaves $\mathcal{F}^p$. The standard algebraic topology techniques identify all these homology theories with the tropical (co)homology.

Proposition 2.15. *There are natural isomorphisms*

$$H_{p,q} \cong H_q(X, \mathcal{F}_p) \quad \textit{and} \quad H^{p,q} \cong H^q(X, \mathcal{F}^p),$$

where on the right hand side are the sheaf-theoretic (co)homology groups.

3. Tropical waves

3.1. **Waves and cowaves.** There is also another collection of sheaves and cosheaves that can be associated to a tropical space X. We still assume X is compact. Let us revisit Definition 2.1 where we considered an embedding $\phi : U \to Y \subset \mathbb{T}^N$ for a point $x \in X$. Let W' be the largest vector subspace of the tangent space $T_{\phi(x)}(\mathbb{T}^\circ_{I(\phi(x))})$ that is contained in the image of every facet of X adjacent to x under ϕ. Clearly, W' is defined over $\mathbb{Z}$ and is contained in the linear span of $\phi(U) \cap \mathbb{T}^\circ_{I(\phi(x))}$ considered

in Definition 2.1. As W' depends only on the face Δ containing x in its relative interior we may write W'_Δ for this space.

In the case Δ is infinite we need to further modify the vector space W'. Namely, we take the quotient of W' by the divisorial directions (cf. Definition 1.3) and denote the resulting quotient vector space by W_Δ.

On the other hand, every face Δ of sedentarity s is contained in a unique *minimal* face Δ'' of sedentarity 0. (We have $\dim \Delta'' = \dim \Delta + s$). Then we set $W''_\Delta := W'_{\Delta''}$.

To summarize, we have $W = W' = W''$ for finite faces of sedentarity 0. For infinite faces the W is the quotient of W' by the divisorial directions. $W'' = W$ for zero sedentarity infinite faces. For infinite faces of positive sedentarity the W'' equals W' of the smallest adjacent face of sedentarity 0.

Definition 3.1. We define $W_k(\Delta)$ as the free abelian group generated by the integer elements of the exterior power $\Lambda^k(W_\Delta)$. Similarly, we define $W''_k(\Delta)$ as the free abelian group generated by the integer elements of the exterior power $\Lambda^k(W''_\Delta)$. We also consider the corresponding dual groups $W^k(\Delta)$ and $W''^k(\Delta)$.

It is also convenient to write $W_k(x) = W_k(\Delta)$ and $W''_k(x) = W''_k(\Delta)$ for every point x in the relative interior of the face Δ, and similar for $W^k(x)$ and $W''^k(x)$.

Note that for a pair $\Delta' \subset \Delta$ of two faces we have the natural homomorphisms

$$\pi : W_k(\Delta') \to W_k(\Delta) \quad \text{and} \quad \pi : W''_k(\Delta') \to W''_k(\Delta) \tag{3.1}$$

as well as the dual ones

$$W^k(\Delta) \to W^k(\Delta') \quad \text{and} \quad W''^k(\Delta) \to W''^k(\Delta').$$

By Proposition 2.14 the coefficient systems W_k and W''_k define constructible sheaves $\mathcal{W}_k$ and $\mathcal{W}''_k$ on X, whereas the W^k and W''^k define cosheaves $\mathcal{W}^k$ and $\mathcal{W}''^k$ for every integer $k \geq 0$.

Proposition 3.2. *For compact tropical space X we have canonical isomorphisms $H^q(X; \mathcal{W}_k) \cong H^q(X; \mathcal{W}''_k)$ and $H_q(X; \mathcal{W}^k) \cong H_q(X; \mathcal{W}''^k)$.*

Proof. Note that we have a surjective homomorphism of sheaves $\mathcal{W}''_k \to \mathcal{W}_k$ which induces a homomorphism of cochain complexes

$$C^\bullet_{cell}(X; \mathcal{W}''_k) \to C^\bullet_{cell}(X; \mathcal{W}_k) \tag{3.2}$$

commuting with the boundary map. Let us prove that all cocycles in the kernel K of (3.2) are coboundaries of other elements of K.

Recall that W_k and W''_k only differ at infinite cells and at the cells of positive sedentarity. Note that an element of K cannot be a cocycle unless its value on each k-face Δ of positive sedentarity is zero. Indeed, the boundary of such cochain cannot vanish on the $(k+1)$-face adjacent to Δ of sedentarity $s(\Delta) - 1$.

Conversely, if a cochain in K takes a non-zero value at an infinite face Δ then this value can be presented as the sum of the boundary divisor directions. This defines a $(k-1)$-dimensional cochain whose coboundary is our cocycle from K.

As all cocycles in K are coboundaries the map $H^q(X;\mathcal{W}_k) \cong H^q(X;\mathcal{W}_k'')$ induced by (3.2) is injective. Also now surjectivity of (3.2) implies the surjectivity of the induced map in homologies. More precisely, we lift a cocycle $\alpha \in C^\bullet_{cell}(X;\mathcal{W}_k)$ to an arbitrary cochain $\alpha'' \in C^\bullet_{cell}(X;\mathcal{W}_k'')$ under (3.2). The boundary $\delta\alpha''$ belongs to K and therefore there exists an element $\gamma'' \in K$ with $\delta\gamma'' = \delta\alpha''$. The cocycle α is an image of the cocycle $\alpha'' - \gamma''$. $\square$

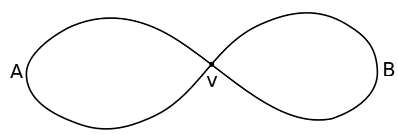
Definition 3.3. Tropical *wave* and *cowave* groups, respectively, are

$$H^q(X;\mathcal{W}_k) \quad \text{and} \quad H_q(X;\mathcal{W}^k). \tag{3.3}$$

Again, we can think of these groups from the sheaf-theoretic point of view or as face-compatible singular (co)homology with coefficients in the systems W_k and W^k.

Example 3.4. Let us consider a tropical genus 2 curve C with a simple double point. The underlying topological space of C is a wedge of two circle, i.e. it is a graph with a single vertex v and two edges that are glued to v, see Figure 3.4.

The tropical structure in the interior of each edge is isomorphic to an open interval of finite length in $\mathbb{R}$ (treated as the tropical torus $\mathbb{T}^\times = \mathbb{T} \smallsetminus \{-\infty\}$). The tropical



structure at the vertex v is such that the four primitive vectors divide into 2 pairs of opposite vectors. This means that the chart at v is given by a map to $\mathbb{R}^2$ such that a neighborhood of v in C goes to the union of coordinate axes and the four primitive vectors near v go to the unit tangent vectors to those axes.

Thus, $\mathcal{F}_1(v) = \mathbb{Z}^2$ and $W_1(v) = 0$. On the other hand every point x in the interior of either edge has the groups $\mathcal{F}_1(x) = \mathbb{Z}$ and $W_1(x) = \mathbb{Z}$. The groups $\mathcal{F}_0(x)$ and $W_0(x)$ are always $\mathbb{Z}$ for any point x. From the two term cell complex one can easily calculate

$$H_0(C;\mathcal{F}_0) \cong \mathbb{Z}, \quad H_0(C;\mathcal{F}_1) \cong \mathbb{Z}, \quad H_1(C;\mathcal{F}_0) \cong \mathbb{Z}^2, \quad H_1(C;\mathcal{F}_1) \cong \mathbb{Z},$$

and

$$H^0(C;\mathcal{W}_0) \cong \mathbb{Z}, \quad H^0(C;\mathcal{W}_1) = 0, \quad H^1(C;\mathcal{W}_0) \cong \mathbb{Z}^2, \quad H^1(C;\mathcal{W}_1) \cong \mathbb{Z}^2.$$

In general the case of $p = 0$ is easy since $\mathcal{F}_0$, $\mathcal{W}_0'$ and $\mathcal{W}_0$ are constants, thus we have the following obvious observation.

Proposition 3.5. *We have $H_{0,q} = H_q(X;\mathcal{W}^0) = H_q(X)$ and $H^{0,q} = H^q(X;\mathcal{W}_0) = H^q(X)$, where $H_q(X)$ and $H^q(X)$ stand for the topological q-dimensional homology and cohomology groups with integer coefficients.*

3.2. **Pairing of $\mathcal{F}$ and $\mathcal{W}$.** The importance of the wave classes stems from their action on the tropical homology via a natural bilinear map

$$\cap : H^s(X;\mathcal{W}_k) \otimes H_q(X;\mathcal{F}_p) \to H_{q-s}(X;\mathcal{F}_{p+k})$$

which we are going to define now. To do this on the chain level we will use the canonical isomorphism $H^s(X;\mathcal{W}_k) \cong H^s(X;\mathcal{W}''_k)$ from Proposition 3.2. Then this map is just the standard cap product between singular chains and cochains coupled with the wedge product on the coefficients $\wedge : \mathcal{W}''_k \otimes \mathcal{F}_p \to \mathcal{F}_{p+k}$.

Precisely, let α be a face-compatible s-cochain with coefficients in $\mathcal{W}''_k$ and $\gamma = \sum \beta\sigma$ be a tropical q-chain with coefficients in $\mathcal{F}_p$. For each singular simplex σ we denote by $\sigma_{0\ldots s}$ its first s-face (spanned by the first $s+1$ vertices of σ) and by $\sigma_{s\ldots q}$ its last $(q-s)$-face. Then we set

$$\alpha \cap \gamma = \sum (\alpha(\sigma_{0\ldots s}) \wedge \beta)\sigma_{s\ldots q}. \tag{3.4}$$

Here we push the value α from the face $\sigma_{0\ldots s}$ to the simplex σ with the sheaf map and then push the value of the result from σ to $\sigma_{s\ldots q}$ using the cosheaf map.

Before showing that this product descends to homology level we need the following straightforward observation.

Lemma 3.6. *Let $\Delta' \prec \Delta$ be a pair of adjacent faces in X. Then the diagram*

$$\begin{tikzcd}
W''_k(\Delta) \otimes \mathcal{F}_p(\Delta) \arrow[r, "\wedge"] \arrow[d, "\iota" near start, shift left=6ex] & \mathcal{F}_{p+k}(\Delta) \arrow[d, "\iota"] \\
W''_k(\Delta') \otimes \mathcal{F}_p(\Delta') \arrow[r, "\wedge"] \arrow[u, "\pi", shift left=-6ex] & \mathcal{F}_{p+k}(\Delta')
\end{tikzcd}$$

is commutative in the sense that for any $\alpha \in W''_k(\Delta')$ and $\beta \in \mathcal{F}_p(\Delta)$ one has $\iota(\pi(\alpha) \wedge \beta) = \alpha \wedge \iota(\beta)$.

Proof. The wedge product is bilinear with respect to inclusion and quotient (in fact, all) homomorphisms between free abelian groups. $\square$

Proposition 3.7. *For each $s \le q$ the cap product* (3.4) *descends to a natural bilinear map in homology*

$$\cap : H^s(X;\mathcal{W}_k) \otimes H_q(X;\mathcal{F}_p) \to H_{q-s}(X;\mathcal{F}_{p+k}).$$

Proof. The statement follows at once from the usual Leibnitz formula

$$(-1)^s\partial(\alpha \cap \gamma) = (\delta\alpha) \cap \gamma + \alpha \cap \partial\gamma.$$

Note that the wedge products in $\delta(\alpha \cap \gamma)$ and $(\partial\alpha) \cap \gamma$ are taken in σ and then pushed to $\mathcal{F}_{p+k}(\sigma_{s\ldots\hat{\imath}\ldots q})$. On the other hand the wedge products in $\alpha \cap \delta\gamma$ are taken in $\sigma_{0\ldots\hat{\imath}\ldots q}$ and then pushed to $\mathcal{F}_{p+k}(\sigma_{s\ldots\hat{\imath}\ldots q})$. But Lemma 3.6 allows us to identify the results. $\square$

3.3. **The group $H^1(X;\mathcal{W}_1\otimes\mathbb{R})$ and deformations of the tropical structure of X.** In this section we assume that X is compact. The cocycles in $C^1_{cell}(X;\mathcal{W}_1'')$ determine infinitesimal deformations of tropical structure in X. To see this we consider a simplicial decomposition of X, i.e. we take the baricentric subdivision (adjusted in sedentary faces as in Figure 2) of a decomposition of X as a polyhedral complex.

Given a face Δ of X we consider its *open star* $\mathrm{St}(\Delta)$, that is the union of the relative interiors of faces containing Δ. Open stars $\mathrm{St}(v)$ of the vertices $v\in X$ provide a finite open covering of X. It will be convenient to consider slightly smaller neighborhoods $U_v\subset\mathrm{St}(v)$, such that we still have a covering

$$X=\bigcup_v U_v,$$

but now with the topological closure of each U_v contained in $\mathrm{St}(v)$.

Then to obtain X one can take the disjoint union $\amalg_v(U_v)$ and identify pairs of points (v,y_v) and $(v',y_{v'})$ whenever $y_v=y_{v'}$. Note that this is an equivalence relation by definition. In particular, it is transitive. Also note that if $(v,y_v)\sim(v',y_{v'})$ then $U_v\cap U_{v'}\neq\emptyset$ and therefore v and v' are connected by an edge $[v,v']$.

One important observation is that $W_1''([v,v'])$ is contained in W_1'' of any strata in $\mathrm{St}([v,v'])$. Thus we can translate any point x in $\mathrm{St}([v,v'])$ by a small scalar multiple of any vector in $W_1''([v,v'])$.

Let $\alpha\in C^1_{cell}(X;\mathcal{W}_1'')$ be a 1-cocycle and choose a small positive number $\epsilon>0$ (the size of deformation).

Proposition 3.8. *The identification $(v,y_v)\sim(v',y_{v'})$ for all pairs $y_v,y_{v'}\in\mathrm{St}([v,v'])$ such that $y_v=y_{v'}-\epsilon\alpha([v,v'])$ defines an equivalence relation on $\amalg_v(U_v)$.*

Proof. First note that $\alpha([v,v'])\in W_1''([v,v'])$, and we consider only the points $y_{v'}\in U_{v'}$ for which the translation $y_{v'}-\epsilon\alpha([v,v'])$ makes sense as a point in $U_{v'}$. The identification is symmetric since $\alpha([v,v'])=-\alpha([v',v])$. It is transitive since α is a cocycle. $\square$

Definition 3.9. The result $X_{\alpha,\epsilon}$ of deformation of X with parameters $\epsilon>0$ and $\alpha\in C^1_{cell}(X;\mathcal{W}_1'')$ is the quotient space of $\amalg_v(U_v)$ by this equivalence relation.

Clearly, the open sets U_v provide a covering of $X_{\alpha,\epsilon}$ with the same local embeddings $\phi_v: U_v\to Y_v\subset\mathbb{T}^{N_v}$. Thus $X_{\alpha,\epsilon}$ is also a tropical space. Furthermore, it has the same *combinatorial type* as X, i.e. it has the same combinatorics of the face structure, but the faces of $X_{\alpha,\epsilon}$ themselves may have different affine structure. E.g. if X is 1-dimensional, the lengths of the edges of X and $X_{\alpha,\epsilon}$ may be different.

We give now another construction of the deformed tropical space $X_{\alpha,\epsilon}$ thinking of α as a Čech cocycle $\alpha\in C^1_{Cech}(X;\mathcal{W}_1'')$ with respect to an open covering $\{U_j\}$ of X. We may assume that each U_j is contained in the U_v, the "shrunk" open stars of vertices for some simplicial subdivision of X. Thus for $x\in U_j\cap U_{j'}$ and small

enough $\epsilon > 0$ the expressions $x + \epsilon\alpha_{jj'}$, where $\alpha_{jj'} \in \mathcal{W}''_1(U_j \cap U_{j'})$ is the value of the cocycle, make sense. Then again we may present X as the quotient space of $\amalg_j(U_j)$ by the equivalence relation identifying the same points in different open sets. The cocycle α defines a deformation of this equivalence relation by replacing $(j, x) \sim (j', x)$ with $(j, x + \epsilon\alpha_{jj'}) \sim (j', x)$. We set $X_{\alpha,\epsilon}$ to be the quotient of $\amalg_j(U_j)$ by this new equivalence relation.

Note that if the Čech cover $\{U_j\}$ is taken to be the open star cover $\{U_v\}$ for some simplicial subdivision of X then it consists of contractible sets and all intersections are contractible open sets and $\mathcal{W}''_1(U_v \cap U_{v'}) = W''_1([v, v'])$. Moreover, the Čech and the simplicial complexes are identical. Thus we see that in this case the Čech formalism gives the same space $X_{\alpha,\epsilon}$ as does the cellular approach. From the Čech point of view, however, it is easier to see independence of the simplicial subdivision of X.

Proposition 3.10. *Two cellular cocycles $\alpha, \alpha' \in C^1_{cell}(X; \mathcal{W}''_1)$ in the same cohomology class define isomorphic tropical spaces $X_{\alpha,\epsilon}$ and $X_{\alpha',\epsilon}$.*

Two Čech cocycles $\alpha, \alpha' \in C^1_{Cech}(X; \mathcal{W}''_1)$ in the same cohomology class for two different simplicial subdivisions define isomorphic tropical spaces $X_{\alpha,\epsilon}$ and $X_{\alpha',\epsilon}$.

Thus we may speak of the tropical space $X_{[\alpha],\epsilon}$ (up to isomorphism) for a homology class $[\alpha] \in H^1(X; \mathcal{W}''_1) \cong H^1(X; \mathcal{W}_1)$ without specifying the subdivision of X.

Proof. By additivity we may assume that $\alpha' - \alpha = \delta\beta$ where $\beta \in C^0_{cell}(X; \mathcal{W}''_1)$ takes a non-zero value on a single vertex $v \in X$ and zero value everywhere else. The constructions of the spaces $X_{\alpha,\epsilon}$ and $X_{\alpha',\epsilon}$ differ in the way of gluing V_v through a parallel translation by a vector in $W''(v)$. The required isomorphism is provided by the composition with this translation.

To see the second statement note that any two Čech coverings have a common refinement. Thus it is enough to show that the constructions agree when a new connected open set $U' = U_0$ is added to an existing covering $\mathcal{U} = \{U_j\}$. We may assume that U' is contained in an open star $\mathrm{St}(v)$ of a vertex v for some simplicial subdivision of X.

Suppose we have a Čech cocycle α for the covering $\mathcal{U}$. Among all U_j which intersect U' we pick one, say U_{i_0}, and set $\alpha_{0i_0} = 0$. If $U' \cap U_j \neq 0$ for some other j, then there is a sequence of open sets $U_{i_0}, U_{i_1}, \dots, U_{i_k} = U_j$ such that $U_{i_s} \cap U_{i_{s-1}} \cap U' \neq \emptyset$. We set

$$\alpha_{0j} = \sum_{s=1}^{k} \alpha_{i_{s-1}i_s}.$$

Since $\mathrm{St}(v)$ is contractible α_{0j} does not depend on the choice of the sequence U_{i_s}. Adding the collection of all such α_{0j} to the existing values α_{ij} defines a cocycle for the refined covering in the same Čech cohomology class as α. Clearly $X_{\alpha,\epsilon}$ does not change after such refinement. $\square$

3.4. **Special cowave classes.** Let Y be a weighted q-dimensional polyhedral pseudocomplex, see Definition 2.8. Then using Proposition 2.11 we get a tropical chain in $C_q(X;\mathcal{F}_q)$. It is a cycle if and only if Y is balanced in the sense of Definition 2.9.

Definition 3.11. A *coweight* of a facet F of Y is a choice of an element $c_F \in W''^m(\Delta_F) \cong \mathbb{Z}$, where Δ_F is the face of X containing F and $m = \dim W''_{\Delta_F}$. We say that Y is *coweighted* if every its facet is prescribed a coweight.

We say that Y is m-pure if $\dim W''(\Delta_F) = m$ for every facet F of Y.

Any facet F of Y sits in a q-dimensional affine space L in a face Δ of X. Recall that the volume element Vol_L is defined up to sign determined by the orientation of F. Thus, Vol_L together with a coweight of F determine an element $\mathrm{Covol}_L \in W''^{(m-q)}(\Delta_F)$ by

$$\lambda \mapsto c_F(\lambda \wedge \mathrm{Vol}_L) \in \mathbb{Z}$$

for $\lambda \in W''_{m-q}(\Delta_F)$.

Thus a triangulation of a coweighted m-pure q-dimensional polyhedral pseudocomplex Y gives rise to a cowave chain in $C_q(X;\mathcal{W}''^{(m-q)})$. (As in the case with special tropical cycles the orientation of F enters the resulting cowave cycle twice.) Such cowave chain is called *special*.

Definition 3.12. A coweighted polyhedral pseudocomplex $Y \subset X$ is called *cobalanced* if the resulting special cowave chain is a cycle. We may refine this notion by saying that Y is cobalanced at $x \in Y$ if x is disjoint from the support of the boundary of the resulting special cowave cochain.

Once an orientation of $W''(\Delta_F)$ is chosen we may identify coweight and weight of F. If Y is m-pure and $x \in \partial F$ is such that $\dim W''(x) = m$ then $W''(x) = W''(\Delta_F)$ and we get the following proposition by converting the coweights to weights with the help of arbitrary orientation of $W''(x)$.

Proposition 3.13. *If Y is m-pure and $x \in \mathrm{Sk}^{q-1}(Y)$ is a generic junction point with $\dim W''(x) = m$ then Y is cobalanced at x if and only if Y is balanced at x.*

In the same time if $\dim W''(x) < m$ then the cobalancing condition is different from the balancing condition. We believe that study of special cowave cycles might be useful, particularly in the context of mirror symmetry.

4. The eigenwave

4.1. **The eigenwave ϕ.** There is a canonical element $\phi \in H^1(X;\mathcal{W}_1 \otimes \mathbb{R})$ for every compact tropical space X. Let us first define it through a canonical singular cocycle $\phi_{sing} \in C^1(X;\mathcal{W}_1 \otimes \mathbb{R})$. Note that the dual space to $W_1(\Delta_\sigma) \otimes \mathbb{R}$ is $W^1(\Delta_\sigma) \otimes \mathbb{R}$. Thus such a cocycle is defined by real values on singular 1-simplices σ with coefficients $\alpha_\sigma \in W^1(\Delta_\sigma) \otimes \mathbb{R}$. Note that α_σ is a constant 1-form on the face Δ_σ containing the interior of σ and thus we can integrate α_σ against σ.

Definition 4.1. We set

$$\phi_{sing}(\alpha_\sigma \sigma) = \int_\sigma \alpha_\sigma. \tag{4.1}$$

Proposition 4.2. *The integral in* (4.1) *is finite and the resulting cochain ϕ_{sing} is a cocycle.*

Proof. If the face Δ_σ is finite then the integral (4.1) is automatically finite. However even if Δ_σ is infinite then the form α_σ vanishes on all divisorial directions in $W'(\Delta_\sigma)$ by the definition of $W(\Delta_\sigma)$ and the image of σ in $W(\Delta_\sigma)$ is always finite.

Consider the coboundary $\delta\phi_{sing}$. Its value on any 2-simplex τ enhanced with a coefficient $\alpha_\tau \in W^1(\Delta_\tau) \otimes \mathbb{R}$ is obtained by integrating the 1-form α_τ against the boundary of the triangle τ. But α_τ is a constant form, therefore it is closed. Note that to evaluate $\int_{\partial\tau} \alpha_\tau$ we might need to pass to a subface with the help of the homomorphisms dual to (3.1) if $\partial\tau$ has edges contained in another combinatorial strata $\Delta'_\tau \subset \partial\Delta_\tau \subset X$. Nevertheless, the value of the integral stays the same as the corresponding homomorphism of forms agrees with the inclusion $\Delta'_\tau \prec \Delta_\tau$. ☐

We will denote by w_σ the functional of integration of 1-forms $\alpha_\sigma \in W^1(\Delta_\sigma) \otimes \mathbb{R}$ along the interval σ. One can think of w_σ as a vector in $W_1(\Delta_\sigma) \otimes \mathbb{R}$ connecting the end points of σ.

Definition 4.3. The homology class $\phi = [\phi_{sing}] \in H^1(X; \mathcal{W}_1 \otimes \mathbb{R})$ is called the *tropical eigenwave*.

Remark 4.4. Below we consider the cap product with ϕ and its powers which was defined for the classes in $H^1(X; \mathcal{W}''_1 \otimes \mathbb{R})$. So in general we will have to use the isomorphism $H^1(X; \mathcal{W}_1) \cong H^1(X; \mathcal{W}''_1)$ to identify ϕ with the class $\phi'' \in H^1(X; \mathcal{W}''_1 \otimes \mathbb{R})$. In case X is compact, however, we will manage to define this cap product on the cycle level with $\phi \in H^1(X; \mathcal{W}_1)$.

Definition 4.5. It is also convenient to define a cellular cochain $\phi_{cell} \in C^1_{cell}(X; \mathcal{W}_1 \otimes \mathbb{R})$. The value of ϕ_{cell} on an oriented finite edge E of the subdivision is the same oriented edge E represented as a vector in the vector space $W(E)$.

Proposition 4.6. *The homology classes of ϕ_{cell} and ϕ_{sing} coincide under identification of the cellular and singular homology of X.*

Proof. The two definitions agree on the singular 1-chains composed of the edges of X. ☐

4.2. **Action of the eigenwave ϕ and its powers on tropical homology.** Note that the k-th cup powers of ϕ are also wave classes $\phi^k \in H^k(X; \mathcal{W}_k \otimes \mathbb{R})$. One can define the ϕ^k on the singular chain level. Namely, let $\sigma \in \Delta_\sigma$ be a k-simplex. Recall that $w_e \in W_1(\Delta_\sigma)$ denotes the integration along an edge e of σ. Then

$$\phi^k_{sing}(\sigma) = w_{\sigma_{01}} \wedge \cdots \wedge w_{\sigma_{k-1,k}} =: w_\sigma \in W_k(\Delta_\sigma) \otimes \mathbb{R}. \tag{4.2}$$

Taking the cap product with $\phi^k = [\phi^k_{sing}]$ gives us homomorphisms:

$$\phi^k \cap : H_q(X; \mathcal{F}_p) \otimes \mathbb{R} \to H_{q-k}(X; \mathcal{F}_{p+k}) \otimes \mathbb{R}. \tag{4.3}$$

In case X is compact we can consider its baricentric subdivision and think of the $H_q(X; \mathcal{F}_p)$ as simplicial or cellular homology groups. The advantage of this is that we can now define the cap product of ϕ^k with cycles in $C^{bar}_q(\mathcal{F}_p \otimes \mathbb{R})$. Below we give two different descriptions of the maps (4.3) on the cycle level depending on the choice of vertex ordering.

Let $\gamma = \sum \beta_\sigma \sigma$ be a cycle in $C^{bar}_q(X; \mathcal{F}_p)$ supported on the q-skeleton of X. Then the coefficients β_σ for all σ lying in the same q-face $\Delta \subset X$ have to be the same (that is we can also think of γ as a cellular cycle). We denote them by β_Δ. On the other hand, the cycle condition says that β_Δ has to be divisible by the divisorial directions in Δ. In particular, the wedge product of β_Δ with any element in $W_k(\Delta) \otimes \mathbb{R}$ gives a well-defined element in $W_{p+k}(\Delta) \otimes \mathbb{R}$.

Let $\Delta \in X$ be a q-cell in the support of the cycle γ. For any *finite* j-dimensional face $\Delta' \prec \Delta$ of the sedentarity $s(\Delta') = s(\Delta)$ we denote by $\hat{\Delta}'_\Delta$ its dual cell in Δ. Namely, $\hat{\Delta}'_\Delta$ is the union of all $(q-j)$-simplices containing the baricenters of Δ and Δ'. We can think of $\hat{\Delta}'_\Delta$ as a simplicial chain. The orientations of the pair Δ' and $\hat{\Delta}'_\Delta$ are taken to agree with the original orientation of Δ.

Choice 1: We label the vertices of each q-simplex σ in the baricentric subdivision $bar(\Delta)$ according to dimension of the largest cells whose baricenters they represent (recall that several faces of Δ of different sedentarity may have the same baricenter). In this case by definition of the cap product on the chain level the cycle $\phi^k_{sing} \cap \gamma_1 \in C^{bar}_{q-k}(X; \mathcal{F}_{p+k})$ is supported on the dual subdivision of the q-skeleton of X.

For every k-face Δ' of Δ let $w_{\Delta'} \in W_k(\Delta) \otimes \mathbb{R}$ denote the volume element associated to Δ' as in (4.2). Clearly, $w_{\Delta'}$ equals the sum of all $w_{\sigma_{0\ldots k}}$ (taken with appropriate signs) for the k-simplices $\sigma_{0\ldots k}$ forming the baricentric triangulation of Δ'. Then one can easily calculate from the definition of the cap product:

$$\phi^k_{sing} \cap \Big(\sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma \Big) = \sum_{\Delta' \prec \Delta} (w_{\Delta'} \wedge \beta_\Delta) \hat{\Delta}'_\Delta, \tag{4.4}$$

where the sum is taken over all k-dimensional faces of Δ. Note that higher sedentary k-faces don't appear in the sum because β_Δ vanishes when pushed to these higher sedentary faces.

Choice 2: Here we label the vertices of each σ in the opposite order to the description 1. That is the baricenters with the smaller numbers correspond to the larger faces. Now the cycle $\phi^k_{sing}(\gamma)_2 \in C^{bar}_{q-k}(X; \mathcal{F}_{p+k})$ is supported on the $(q-k)$-skeleton of X.

For every $(q-k)$-face Δ' of Δ let $\hat{w}_{\Delta'} \in W_k(\Delta) \otimes \mathbb{R}$ denote the polyvector corresponding to the integration along the chain $\hat{\Delta}'_\Delta$. Note that the faces $\sigma_{k\ldots q}$ lie in the $(q-k)$-faces of Δ. The polyvectors $w_{\sigma_{0\ldots k}}$ sum to $\hat{w}_{\Delta'}$ for those simplices

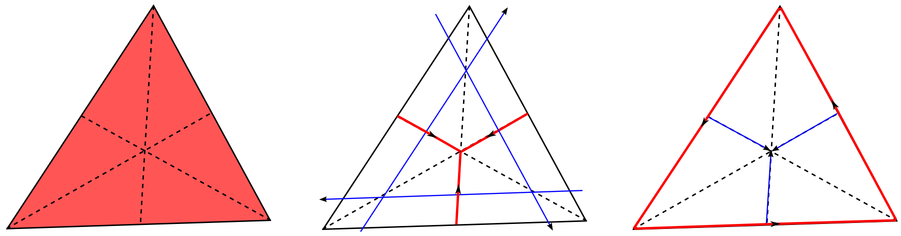


FIGURE 4. The two descriptions of the wave action on a 2-cell σ. The support of $\phi_{sing} \cap \sigma$ is red and the framing is blue.

$\sigma \in bar(\Delta)$ whose faces $\sigma_{k\dots q}$ give the same simplex in $bar(\Delta')$. Then again from the definition of the cap product we can write:

$$\phi^k_{sing} \cap \Big(\sum_{\sigma \in bar(\Delta)} \beta_\Delta \sigma \Big) = \sum_{\Delta' \prec \Delta} (\hat{w}_{\Delta'} \wedge \beta_\Delta) \Delta', \tag{4.5}$$

where the sum is taken now over all $(q-k)$-dimensional faces of Δ.

Conjecture 4.7. *Let X be a smooth compact tropical variety. Then for $q \geq p$*

$$\phi^{q-p} \cap : H_q(X; \mathcal{F}_p) \otimes \mathbb{R} \to H_p(X; \mathcal{F}_q) \otimes \mathbb{R}$$

is an isomorphism.

We will prove the conjecture in the realizable case in Section 6 though we believe that realizability assumption is not necessary. Certain amount of smoothness, on the other hand, is essential. In the non-smooth case even the ranks of $H_q(X; \mathcal{F}_p)$ and $H_p(X; \mathcal{F}_q)$ may not agree. A simple example is provided by the nodal genus 2 curve (see Example 3.4).

The action of the eigenwave ϕ is especially easy on special tropical (p,p)-classes.

Theorem 4.8. *If $\gamma \in H^{special}_{p,p}(X)$ then $\phi \cap \gamma = 0$.*

Proof. Any vector parallel to a simplex of a special tropical cycle turns to zero after the wedge product with the volume element defined by that simplex. Thus we get zero on the chain level if we represent γ by a special tropical cycle and ϕ by ϕ_{sing}. $\square$

5. INTERMEDIATE JACOBIANS

5.1. **Tropical tori.** Let V be a g-dimensional real vector space containing two lattices Γ_1, Γ_2 of maximal rank, that is $V \cong \Gamma_{1,2} \otimes \mathbb{R}$. Suppose we are given an isomorphism $Q : \Gamma_1 \to \Gamma_2^*$, which is symmetric if thought of as a bilinear form on V.

Definition 5.1. The torus $J = V/\Gamma_1$ is the *principally polarized tropical torus* with Q being its polarization. The tropical structure on X is given by the lattice Γ_2. If, in addition Q is positive definite, we say that J is an *abelian variety.*

Remark 5.2. The map $Q : \Gamma_1 \to \Gamma_2^*$ provides an isomorphism of $J = V/\Gamma_1$ with the tropical torus V^*/Γ_2^*. The tropical structure on the latter is provided by the lattice Γ_1^*.

Remark 5.3. The above data $(V, \Gamma_1, \Gamma_2, Q)$ is equivalent to a non-degenerate real-valued quadratic form Q on a free abelian group $\Gamma_1 \cong \mathbb{Z}^g$. The other lattice $\Gamma_2 \subset V := \Gamma_1 \otimes \mathbb{R}$ is defined as the dual lattice to the image of Γ_1 under the isomorphism $V \to (V)^*$ given by Q.

Let us take the free abelian group $\Gamma_1 = H_q(X; \mathcal{F}_p) \cong \mathbb{Z}^g$ with $p+q = \dim X$, and $p \leq q$. We define the tropical intermediate Jacobian as the torus above together with a symmetric bilinear form Q on $H_q(X; \mathcal{F}_p)$.

The form Q is a certain intersection product on tropical cycles which we define in two ways. The first definition is manifestly symmetric while the second definition descends to homology. And then we show that the two definitions are equivalent.

Unfortunately we are not able to show in this paper that the form is non-degenerate, though we believe that under certain smoothness and compactness conditions this should be true (cf. Conjecture 5.16).

5.2. **Intersection product.** Let X be a compact tropical space of dimension n. For a singular simplex σ we denote its relative interior by $int(\sigma)$ as well as its image in X.

Definition 5.4. We say that a tropical chain $\sum \beta_\sigma \sigma \in C_q(X; \mathcal{F}_p)$ is *transversal to the face structure of* X (or, simply, transversal) if for any simplex σ and any face $\tau \prec \sigma$ we have

- $int(\tau) \cap \mathrm{Sk}^{k-1}(X) = \emptyset$, if $\tau \prec \sigma$ has codimension k;
- if τ lies in a sedentary face of X then β_σ is divisible by all corresponding divisorial vectors of that face.

Definition 5.5. We say that two transversal tropical chains $\sum \beta_{\sigma'} \sigma' \in C_{q'}(X; \mathcal{F}_{p'})$ and $\sum \beta_{\sigma''} \sigma'' \in C_{q''}(X; \mathcal{F}_{p''})$ form a *transversal pair* if for every pair of simplices σ', σ'' from these chains and any choice of their faces $\tau' \prec \sigma', \tau'' \prec \sigma''$ with $\Delta_{\tau'} = \Delta_{\tau''} = \Delta$ their interiors $int(\tau'), int(\tau'')$ are transversal in the usual sense as smooth maps to Δ.

Remark 5.6. If a pair of simplices σ', σ'' from the transversal pair have non-empty intersection then all three submanifolds $\sigma', \sigma'', \sigma' \cap \sigma''$ are supported on the same facet $\Delta_{\sigma' \cap \sigma''}$ of X (and on no smaller face). The oriented triple $\sigma', \sigma'', \sigma' \cap \sigma''$ determines an orientation of $\Delta_{\sigma' \cap \sigma''}$, thus an integral volume element $\mathrm{Vol}_{\Delta_{\sigma' \cap \sigma''}}$ as well as its dual volume form $\Omega_{\Delta_{\sigma' \cap \sigma''}}$. By transversality, $\sigma' \cap \sigma''$ has dimension $q' + q'' - n$. We can choose a singular chain $\sum \tau$ representing its relative fundamental class agreeing with the orientation of $\sigma' \cap \sigma''$.

Let $\gamma' = \sum \beta_{\sigma'}\sigma' \in C_{q'}(X;\mathcal{F}_{p'})$ and $\gamma'' = \sum \beta_{\sigma''}\sigma'' \in C_{q''}(X;\mathcal{F}_{p''})$ be a transversal pair of tropical chains. We define the following bilinear product with values in the cowave chains:

$$\gamma' \cdot \gamma'' = \sum_{\tau \subset \sigma' \cap \sigma''} \Omega_{\Delta_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''}) \cdot \tau \in C_{q'+q''-n}(X; \mathcal{W}^{n-p'-p''}). \tag{5.1}$$

Remark 5.7. Note that $\gamma' \cdot \gamma''$ has no support on infinite simplices $\tau \subset \sigma' \cap \sigma''$ since its divisorial directions divide both $\beta_{\sigma'}$ and $\beta_{\sigma''}$.

Remark 5.8. If $q' + q'' < n$ or $p' + p'' > n$ then $\gamma' \cdot \gamma'' = 0$ for dimensional reasons. In what follows we will tacitly assume this is not the case.

Our goal will be to show that this product descends to homology once the tropical space X is sufficiently nice. Namely, we'll require that X is a *tropical manifold*. For this we need to recall matroids and tropical spaces associated to them (cf. e.g. [AK06], [Sh12], [MR12]).

A matroid $M = (M, r)$ is a finite set M together with a rank function $r : 2^M \to \mathbb{Z}_{\geq 0}$ such that we have the inequalities $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ and $r(A) \leq |A|$, where $|A|$ is the number of elements in A, for any subsets $A, B \subset M$ as well as the inequality $r(A) \leq r(B)$ whenever $A \subset B$. Subsets $F \subset M$ such that $r(A) > r(F)$ for any $A \supset F$ are called *flats* of M of rank $r(F)$. Matroid M is *loopless* if $r(A) = 0$ implies $A = \emptyset$.

The Bergman fan of a loopless matroid M is a non-compact tropical space $Y_M \subset \mathbb{R}^{|M|-1}$ constructed as follows. Choose $|M|$ integer vectors $e_j \subset \mathbb{Z}^{|M|-1} \subset \mathbb{R}^{|M|-1}$, $j \in M$ such that $\sum\limits_{j \in M} e_j = 0$ and any $|M| - 1$ of these vectors form a basis of $\mathbb{Z}^{|M|-1}$. To any flat $F \subset M$ we associate a vector

$$e_F := \sum_{j \in F} e_j \in \mathbb{R}^{|M|-1}.$$

E.g, $e_M = e_\emptyset = 0$, but $e_F \neq 0$ for any other (proper) flat F. To any flag of flats $F_{i_1} \subset \cdots \subset F_{i_k}$ we associate a convex cone generated by $e_{F_{i_j}}$. We define Y_M to be the union of such cones, which is, clearly, an $(r(M)-1)$-dimensional integral simplicial fan. It is easy to check (cf. [AK06]) that it satisfies the balancing condition, so that Y_M is a tropical space, called the *Bergman fan* of M.

The matroid M is called *uniform* if $r(A) = |A|$ for any $A \subset M$. Note that the Bergman fan of a uniform matroid is a complete unimodular fan in $\mathbb{R}^{|M|-1}$ with $|M|$ maximal cones.

Definition 5.9. A tropical space X is called a *smooth*, or a *tropical manifold*, if all its charts ϕ_α are open embeddings to $Y_M \times \mathbb{T}^s \subset \mathbb{T}^{|M|-1} \times \mathbb{T}^s$ for some loopless matroid M and a number $s \geq 0$. (Here s is the maximal sedentarity in this chart and $n = r(M) - 1 + s$ is the dimension of our tropical manifold X.)

Tropical manifolds can be thought of as tropical spaces without points of multiplicity greater than 1, see [MR12], thus we use the term *smooth*. In smooth tropical spaces we can deform all cycles to a transverse position.

The next lemma says that we can move a tropical cycle γ off a face E of X, if it intersects it in higher than expected dimension, not changing it outside the open star $\mathrm{St}(E)$.

Lemma 5.10. *Let $\gamma \in C_q(X, \mathcal{F}_p)$ be a (singular) tropical cycle in a tropical n-dimensional manifold X and E be its l-face of sedentarity s. Then there exists a cycle $\gamma' = \sum \beta_\sigma \sigma \in C_q(X, \mathcal{F}_p)$ homologous to γ and such that for any $(q-k)$-face τ of a simplex σ we have $int(\tau) \cap E = \emptyset$, i.e. τ is not supported on E whenever $k + l < n$. In addition, γ' satisfies to the following properties:*

- *$\gamma \cap (X \smallsetminus \mathrm{St}(E)) = \gamma' \cap (X \smallsetminus \mathrm{St}(E))$,*
- *the chain $\gamma - \gamma'$ is the boundary of a tropical $(q+1)$-chain supported in $\mathrm{St}(E)$,*
- *if E has positive sedentarity then any simplex σ such that $\sigma \cap E \neq \emptyset$ has the coefficient β_σ divisible by all divisorial vectors corresponding to E.*

Proof. First let us consider the case $s = 0$. Then we can replace X with the Bergman fan for some loopless matroid M. Clearly any matroid M contains a uniform submatroid $M_0 \subset M$ of the same rank $r(M)$ (by submatroid we mean a subset with the restriction of the rank function). Thus we have a sequence $M_0 \subset \cdots \subset M_{|M|-r(M)} = M$ of submatroids of M such that M_{j+1} is obtained from M_j by adding one element ϵ_{j+1}. We may form a matroid H_j of rank $r(M_j) - 1$ by setting a new rank function r_{H_j} on M_j, $r_{H_j}(A) = r_M(A \cup \epsilon_{j+1}) - 1$ for $A \subset M_j$.

The fan $Y_{M_{j+1}} \subset \mathbb{R}^{|M_j|}$ maps to the fan $Y_{M_j} \subset \mathbb{R}^{|M_j|-1}$ by projection along the coordinate corresponding to the element ϵ_{j+1}. If the matroid H_j has loops this map

$$\tau_j : Y_{M_{j+1}} \to Y_{M_j}$$

is an isomorphism. Otherwise note that the Bergman fan Y_{H_j} is a subfan of Y_{M_j}. Also we denote by $Y'_{M_{j+1}}$ the subfan of $Y_{M_{j+1}}$ containing only those cones whose corresponding flags do *not* have two flats differing just by ϵ_{j+1}. Then $\tau_j : Y'_{M_{j+1}} \to Y_{M_j}$ is a 1-1 map linear on the cones, cf. [Sh12]. Indeed, τ_j contracts precisely those cones of $Y_{M_{j+1}}$ which are parallel to $e_{\epsilon_{j+1}}$.

For Y_{M_0} the lemma is trivial as the coefficients $\mathcal{F}_p = \Lambda^p \mathbb{Z}^{r(M)-1}$ are constant on all strata and we may deform γ by an arbitrary vector field in $\mathbb{R}^{r(M)-1}$ (subdividing simplices in γ if needed to keep the chain tropical). Inductively we suppose that our lemma holds for Y_{M_j} and the matroid H_j is loopless and then prove that the lemma holds for $Y_{M_{j+1}}$.

We denote by $\mathrm{St}(e_{\epsilon_{j+1}})$ the complement of $Y'_{M_{j+1}}$ in $Y_{M_{j+1}}$. It really is the open star of $e_{\epsilon_{j+1}}$ but in the coarsest face structure of $Y_{M_{j+1}}$. Note that $\mathrm{St}(e_{\epsilon_{j+1}}) \cong Y_{H_j} \times \mathbb{R}$.

If $E \subset \mathrm{St}(e_{\epsilon_{j+1}})$ then we may use the inductive assumption for projections to Y_{H_j} (as it has smaller dimension) together with deformation along a generic vector field parallel to $e_{\epsilon_{j+1}}$.

In the remaining case we have $\dim(\tau_j(E)) = \dim(E) = l$ and E is contained in in $Y'_{M_{j+1}}$. Consider singular q-simplices from γ with the interior mapped to $\mathrm{St}(E)$ and such that their closures intersect E. These simplices form a chain γ_E which can be considered as a relative cycle modulo its boundary $\partial\gamma_E$. We have $\partial\gamma_E \cap E = \emptyset$. Furthermore, $\tau_j(\partial\gamma_E)$ is a $(q-1)$-cycle in the $(n-1)$-dimensional tropical manifold Y_{H_j}. By induction on dimension we may assume that $\tau_j(\partial\gamma_E) \cap \mathrm{St}(e_{\epsilon_{j+1}})$ can be deformed in Y_{H_j} to a cycle with simplices without faces of dimension larger than $q-n+l$ whose relative interiors are contained in E. As $\mathrm{St}(e_{\epsilon_{j+1}}) \cong Y_{H_j} \times \mathbb{R}$ such deformation lifts to $Y_{M_{j+1}}$ and can be extended to a deformation of γ in $Y_{M_{j+1}}$.

By the induction on j there exists a tropical chain $b_j \in C_{q+1}(Y_{M_j}; \mathcal{F}_p)$ such that the relative interiors of k-faces of singular simplices of $\gamma'_j = \partial B_j - \tau_j(\gamma)$ are disjoint from E. This assumption holds for any face structure on Y_{M_j}, in particular for the one compatible with Y_{H_j}. Then the relative interiors of all q-dimensional simplices are disjoint from Y_{H_j} and we can form $\tilde{b}_j \in C_{q+1}(Y_{M_{j+1}}; \mathcal{F}_p)$ and $\tilde{\gamma}'_j \in C_{q+1}(Y_{M_{j+1}}; \mathcal{F}_p)$ by applying $\tau_j^{-1}|_{Y_{M_j} \smallsetminus Y_{H_j}}$ to b_j and γ'_j. Note that $\partial\tilde{b}_j - \gamma - \tilde{\gamma}'_j$ must have the coefficients vanishing under τ_j, even though generated from the facets of $Y'_{M_{j+1}}$. Such coefficients must be supported on $\mathrm{St}(e_{\epsilon_{j+1}})$ and thus we may apply the same reasoning as in the case of $E \subset \mathrm{St}(e_{\epsilon_{j+1}})$.

Finally, let us now consider the sedentary case of $Y_M \times \mathbb{T}^s$ with $s > 0$. It suffices to show that γ can be deformed to γ' so that no (q-dimensional) simplex σ of γ' intersects $Y_M \times \{-\infty\}$ (where $\{-\infty\} \in \mathbb{T}^s$ is the origin, i.e. the point of sedentarity s) along a face of codimension smaller than s. In $Y_M \times \mathbb{R}^s$ the coefficient cosheaf $\mathcal{F}_p$ splits to $\sum\limits_{j=0}^{p} \mathcal{F}_j^{Y_M} \otimes \Lambda^{p-j}(\mathbb{R}^s)$. On $\mathbb{R}^s$ the divisorial directions of $\mathbb{T}^s$ provide the coordinates x_j as well as the integral polyvector $V = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_s}$. Accordingly, we have a decomposition $\gamma = \gamma_0 + \gamma^V$, where Z_0 has coefficients in $\sum\limits_{j=0}^{s-1} \mathcal{F}_j^{Y_M} \otimes \Lambda^{p-j}(\mathbb{R}^s)$.

We have $\gamma_0 = \sum\limits_J \gamma_J$ where $J = \{j_1 < \cdots < j_l\} \subset \{1 < \cdots < n\}$ runs over all possible collections of $l < s$ indices and γ_J has coefficients in $\mathcal{F}_{p-l}^{Y_M} \otimes (\frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_l}})$. Thus, there exists a coordinate $j \notin J$ and we may push γ_J from E with the help of a vector field parallel to x_j. Note that γ_J remains a cycle after such deformation as $\frac{\partial}{\partial x_j}$ is not present in the coefficients of γ_J.

The cycle γ_V has coefficients in $\mathcal{F}_{p-s}^{Y_M} \otimes V$ and thus can be interpreted as a relative cycle modulo $\partial\mathbb{T}^s = \mathbb{T}^s \smallsetminus \mathbb{R}^s$ with coefficients in $\mathcal{F}_{p-s}^{Y^M}$ (as V vanishes on $\partial\mathbb{T}^s$ and constant otherwise) and $(T^s, \partial T_s)$ is homeomorphic to the pair $\mathbb{R}^{s-1} \times (\mathbb{R}_{\geq 0}, \{0\})$ of a half-space and its boundary. Thus γ_V may be deformed to a product (after simplicial

subdivision) of the relative fundamental cycle in the s-dimensional half-space with some $(q-s)$-dimensional singular cycle. In particular, no resulting q-simplex can intersect E along a face of codimension less than s. $\square$

Remark 5.11. Let $\Sigma = \bigcup \sigma$ be an integral polyhedral fan (with its cones σ oriented). Then using the inclusion homomorphisms (2.1) we can form the complex $C_k^{(p)} := \oplus_{\dim \sigma = k} \mathcal{F}_p(\sigma)$. In case Σ is a matroidal fan the statement of Lemma 5.10 is equivalent to that the complex $C_\bullet^{(p)}$ has only the highest homology.

Corollary 5.12. *Let X be a tropical manifold. Then*

(1) *Every class in $H_q(X;\mathcal{F}_p)$ is represented by a transversal cycle.*
(2) *Every pair of classes in $H_{q'}(X;\mathcal{F}_{p'})$ and $H_{q''}(X;\mathcal{F}_{p''})$ is represented by a transversal pair of cycles.*
(3) *If γ'_1, γ'_2 are two cycles which represent the same class in $H_{q'}(X;\mathcal{F}_{p'})$ and both form transversal pairs with a cycle $\gamma'' \in C_{q''}(X;\mathcal{F}_{p''})$, then there is $b \in C_{q'+1}(X;\mathcal{F}_{p'})$ which form a transversal pair with γ'', and such that $\partial b = \gamma'_1 - \gamma'_2$.*

Proof. We may start from any tropical cycle and deform it to a position transversal to the face structure by applying Lemma 5.10 face by face starting from 0-dimensional faces and then higher-dimensional faces. (Note that open star of any face is disjoint from all faces of smaller or equal dimension). Suppose that we have two cycles transversal to the face structure of X. As the relative interior of any face E is a manifold we can make interiors of faces of the simplices from these cycles transversal in E by a small deformation with the help of the usual Sard's theorem. This deformation extends to a small deformation in $\mathrm{St}(E)$. Making this procedure face by face in the order of non-decreasing dimension we make any pair of cycles transversal. A similar argument applies to the relative cycle in the last statement of the corollary. $\square$

If $p' + p'' + q' + q'' = 2n$ we can give a numerical value to the product $\gamma' \cdot \gamma''$ by integrating the $(n - p' - p'')$-form $\Omega_{\Delta_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''})$ over the $(q' + q'' - n)$-simplex τ. Indeed, since $\beta_{\sigma'} \wedge \beta_{\sigma''}$ is divisible by all divisorial directions corresponding to sedentary faces of $\tau = \sigma' \cap \sigma''$, the integration can be carried over in the quotient space to those (infinite) coordinates, thus giving a finite answer. Thus we define

$$\int \gamma' \cdot \gamma'' := \sum_{\tau \subset \sigma' \cap \sigma''} \int_\tau \Omega_{\Delta_\tau}(\beta_{\sigma'} \wedge \beta_{\sigma''}) \in \mathbb{R}. \tag{5.2}$$

The most interesting case to us is when $p' + q' = p'' + q'' = n$. Assuming $q' + q'' \geq n$ we can use the eigenwave action on one of the cycles in the pair to make them of complementary dimensions, after which the integration becomes just summing over

the intersection points

$$\langle \gamma', \gamma'' \rangle := \int \gamma' \cdot \gamma'' = \sum_{x \in |\gamma'| \cap |\gamma''|} \Omega_x(\beta'_x \wedge \beta''_x),$$

where β'_x, β''_x are the coefficients at σ', σ'' for their intersection points $x \in \sigma' \cap \sigma''$.

Proposition 5.13. *Let $\gamma' = \sum \beta_{\sigma'}\sigma' \in C_{q'}(X; \mathcal{F}_{p'})$ and $\gamma'' = \sum \beta_{\sigma''}\sigma'' \in C_{q''}(X; \mathcal{F}_{p''})$ be a transversal pair of tropical cycles with $p' + q' = p'' + q'' = n$ and $q' + q'' \geq n$. Let $k := q' - p'' = q'' - p' \geq 0$. Then*

$$\langle \phi^k \cap \gamma', \gamma'' \rangle = \int \gamma' \cdot \gamma''.$$

Proof. First we need a representative of the cycle $\phi^k \cap \gamma'$ such that it still forms a transversal pair with γ''. We fix first and second baricentric subdivisions of the simplices σ' in γ'. Then by transversality of γ'' we can assume that the intersection of each σ' with γ'' is supported on the star skeleton of σ'. That is $\sigma' \cap |\gamma''|$ consists of the k-simplices of the first baricentric subdivision of σ' spanned by the baricenters of the $q' - k, \dots, q'$-dimensional faces τ of σ'. In general we label the k-simplices in the first baricentric subdivision of σ' by the flags of its faces $(\tau_0 \prec \cdots \prec \tau_k)$.

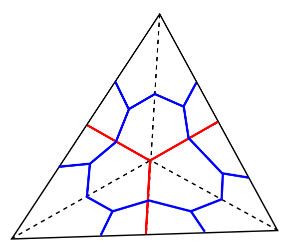


FIGURE 5. Intersection in σ': $|\gamma''|$ (in red), $|\phi^k \cap \gamma'|$ (in blue).

Then the result of the wave action (4.4) from Choice 1 on $\beta_{\sigma'}\sigma'$ gives the following chain (see Fig. 5.2)

$$\sum_{\tau_0 \prec \cdots \prec \tau_k} (w_{\tau_0 \prec \cdots \prec \tau_k} \wedge \beta_{\sigma'})(\tau_0 \widehat{\prec \cdots \prec} \tau_k),$$

where $w_{\tau_0 \prec \cdots \prec \tau_k} \in W_k(\Delta_{\sigma'})$ is the polyvector associated to the simplex $(\tau_0 \prec \cdots \prec \tau_k)$, and $(\tau_0 \widehat{\prec \cdots \prec} \tau_k)$ is its star dual in the second baricentric subdivision (cf. definition in Section 4.2). When intersected with γ'' only the simplices $(\tau_0 \prec \cdots \prec \tau_k)$ with maximal dimensional flags enter and we see that the result coincides with the definition of $\int \gamma' \cdot \gamma''$. $\square$

Recall that we assume out tropical space X is smooth. In particular, if Δ is a codimension 1 face of X the fan at this face $\Sigma(\Delta)$ modulo linear span of Δ is matroidal. That is, the generating primitive vectors $v_1, \dots, v_k$ have just one linear relation among them $\sum_{i=1}^k v_i = 0$ modulo Δ.

Proposition 5.14. *Let X be smooth. Then the intersection product $\langle\ ,\ \rangle$ on cycles descends to a pairing on homology $H_q(X;\mathcal{F}_p)\otimes H_p(X;\mathcal{F}_q)\to\mathbb{R}$.*

Proof. Suppose that we have two homologous cycles $\gamma'_1\in C_q(X;\mathcal{F}_p)$ and $\gamma'_2\in C_q(X;\mathcal{F}_p)$. Let $b\in C_{q+1}(X;\mathcal{F}_p)$ be the connecting chain, i.e. $\partial b=\gamma'_1-\gamma'_2$. According to Corollary 5.12 we can assume that all three γ'_1,γ'_2 and b form transversal pairs with a cycle $\gamma''\in C_p(X;\mathcal{F}_q)$.

It is clear that $\partial(b\cdot\gamma'')$ coincides with the $\gamma'_1\cdot\gamma''-\gamma'_2\cdot\gamma''$ on the interiors of the maximal faces of X. Thus it is enough to show that $b\cdot\gamma''$ has no boundary on codimension 1 faces of X.

Let Δ be such a face and let $\Delta_1,\dots\Delta_k$ be the adjacent maximal faces at Δ. We choose $v_1,\dots,v_k$, the corresponding primitive vectors such that $\sum_{i=1}^k v_i=0$ (not just modulo Δ). Let x be a point in the relative interior of Δ where b intersects γ'', and let $\tau_1,\dots,\tau_k$ be the intervals in the support of $b\cdot\gamma''$ adjacent to x. Each τ_i lies in Δ_i. Let $\beta'_i\in\mathcal{F}_p(\Delta_i)$ and $\beta''_i\in\mathcal{F}_q(\Delta_i)$ be the coefficients of the simplices of b and of γ'', respectively, which intersect at the τ_i.

Since γ'' is a cycle, we have $\sum_i\beta''_i=0$. We can write each

$$\beta''_i=v_i\wedge\bar\alpha''_i+\alpha''_i,$$

where $\bar\alpha''_i\in W_{q-1}(\Delta)$ and $\alpha''_i\in W_q(\Delta)$. Then $\sum_i\beta''_i=0$ together with the only linear dependence $\sum v_i=0$ among the v_i's implies that

$$\sum_{i=1}^k\alpha''_i=0\quad\text{ and }\quad\bar\alpha''_1=\cdots=\bar\alpha''_k=:\bar\alpha''.$$

Similarly, $\sum_i\beta'_i=0$ since ∂b cannot have support at x. Hence we can write

$$\beta'_i=v_i\wedge\bar\alpha'+\alpha'_i,$$

with $\sum\alpha'_i=0$, $\alpha'_i\in W_p(\Delta)$ and $\bar\alpha'\in W_{p-1}(\Delta)$. Note that in the product

$$\beta'_i\wedge\beta''_1=(v_i\wedge\bar\alpha'+\alpha'_i)\wedge(v_i\wedge\bar\alpha''+\alpha''_i)=v_i\wedge(\bar\alpha'\wedge\alpha''_i+\alpha'_i\wedge\bar\alpha'')$$

only the cross terms survive. Now we are ready to evaluate $\partial(b\cdot\gamma'')$ at x:

$$\begin{aligned}\sum_i\Omega_{\Delta_i}[v_i\wedge(\bar\alpha'\wedge\alpha''_i+\alpha'_i\wedge\bar\alpha'')]&=\sum_i\Omega_\Delta(\bar\alpha'\wedge\alpha''_i+\alpha'_i\wedge\bar\alpha'')\\&=\Omega_\Delta(\bar\alpha'\wedge\sum_i\alpha''_i+\sum_i\alpha'_i\wedge\bar\alpha'')=0.\end{aligned}$$

□

Finally we restrict to the case when both γ',γ'' are cycles in $C_q(X;\mathcal{F}_p)$ with $p+q=n$. Then $\gamma'\cdot\gamma''=\gamma''\cdot\gamma'$. Indeed, assuming the orientation of τ is chosen, taking the product in the opposite order will result in the change of orientation of Δ according to the parity of p. On the other hand this parity will also affect the coefficients product: $\beta'\wedge\beta''=(-1)^p\beta''\wedge\beta'$, both effects cancel in $\Omega_{\Delta_\tau}(\beta'\wedge\beta'')$.

This observation combined with Propositions 5.13 and 5.14 lead to following final statement.

Theorem 5.15. *Let X be compact and smooth. The product on cycles* (5.2) *descends to a symmetric bilinear form on $H_q(X;\mathcal{F}_p)$ for any $p+q=n$.*

Conjecture 5.16. *This form is non-degenerate.*

6. Appendix: Konstruktor and the eigenwave action in the realizable case

6.1. **Tropical limit and the Steenbrink-Illusie spectral sequence.** Suppose X is the tropical limit of a complex projective one-parameter degeneration $\mathcal{X} \to \Delta^*$. We say X is smooth if it is locally matroidal and meets the boundary properly (see [MR12]). Smoothness of X and the stable reduction theorem [KKMS73] allow us assume the following (see details in [IKMZ12]).

- X is unimodularly triangulated. This means that the finite cells are unimodular simplices and the infinite cells are products of unimodular simplices and unimodular cones spanned by the extremal brane vectors.
- The finite part of X is identified with the dual Clemens complex of the degeneration with simple normal crossing central fiber $Z = \cup Z_\alpha$. This means that the components of Z are labelled by vertices of zero sedentarity and their intersections $Z_{\alpha_0} \cap \cdots \cap Z_{\alpha_k} =: Z_\Delta$ are labelled by (finite) simplices $\Delta = \{\alpha_0 \ldots \alpha_k\}$ of X of zero sedentarity.
- Each intersection Z_Δ is a blow up of $\mathbb{P}^{n-k}$, and for $\Delta' \prec \Delta$ the inclusions $Z_\Delta \hookrightarrow Z_{\Delta'}$ are linear maps.

Theorem 6.1. *Let X be a realizable smooth projective tropical variety. Then for $q \geq p$*
$$\phi^{q-p} : H_q(X;\mathcal{F}_p) \to H_p(X;\mathcal{F}_q)$$
is an isomorphism.

Note that we don't have to tensor with $\mathbb{R}$ in this algebraic setting, since the eigenwave is itself integral, that is a class in $H^1(X, \mathcal{W}_1)$. We will prove the theorem by comparing the eigenwave action with the classical monodromy action on $H_k(X_t, \mathbb{C})$, where X_t is a general fiber in $\mathcal{X}$.

Notations:

- Δ or Δ' will always denote a finite face of X of sedentarity 0, in particular, a simplex.
- $H_{2l}(\Delta)[-r] = H_{2l}(Z_\Delta, \mathbb{Z})$, Tate twisted by $[-r, -r]$.
- $H_{2l}(k)[-r] = \oplus H_{2l}(\Delta)[-r]$, where Δ runs over all k-simplices in X as above.

First we recall the classical spectral sequence which calculates the limiting mixed Hodge structure of the family $\mathcal{X}$ (see, e.g. [PS08], Chapter 11). This spectral

sequence (from now on referred to as the Steenbrink-Illusie's, or SI for short) has the first term

$$E_1^{r,k-r} = \bigoplus_{i \geq \max\{0,r\}} H_{k+r-2i}(2i-r)[r-i],$$

and it degenerates at E_2 abutting to homology of the smooth fiber X_t of $\mathcal{Z}$ with the monodromy weight filtration.

Since all strata in Z are blow ups of projective spaces, the odd rows in Steenbrink-Illusie's E_1 vanish. Removing those and making shifts in the even rows we relabel the terms by

$$\tilde{E}_1^{q,p} := E_1^{q-p,2p} = \bigoplus_{i \geq \max\{0,q-p\}} H_{2q-2i}(2i+p-q)[q-p-i].$$

The first differential $d = d' + d''$ consists of the map d' induced by strata inclusion and the Gysin map d'':

$$\begin{aligned} d' &: H_{2l}(k)[-r] \to H_{2l}(k-1)[-r] \\ d'' &: H_{2l}(k)[-r] \to H_{2l-2}(k+1)[-r-1]. \end{aligned}$$

For reader's convenience we write the beginning of the $\tilde{E}_1$ term:

$$\begin{tikzcd}
H_0(4)[-4] & & & & \\
H_0(3)[-3] & \begin{matrix} H_0(4)[-3] \\ \oplus H_2(2)[-2] \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & & & \\
H_0(2)[-2] & \begin{matrix} H_0(3)[-2] \\ \oplus H_2(1)[-1] \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & \begin{matrix} H_0(4)[-2] \\ \oplus H_2(2)[-1] \\ \oplus H_4(0) \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & & \\
H_0(1)[-1] & \begin{matrix} H_0(2)[-1] \\ \oplus H_2(0) \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & \begin{matrix} H_0(3)[-1] \\ \oplus H_2(1) \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & \begin{matrix} H_0(4)[-1] \\ \oplus H_2(2) \end{matrix} \arrow[l, "d"'] \arrow[ul, "\nu"'] & \\
H_0(0) & H_0(1) \arrow[l, "d"'] \arrow[ul, "\nu"'] & H_0(2) \arrow[l, "d"'] \arrow[ul, "\nu"'] & H_0(3) \arrow[l, "d"'] \arrow[ul, "\nu"'] & H_0(4) \arrow[l, "d"'] \arrow[ul, "\nu"']
\end{tikzcd}$$

The monodromy operator $\nu = \frac{1}{2\pi i} \log N$ acts along the diagonals by the Tate twist isomorphism $H_{2l}(k)[-r] \to H_{2l}(k)[-r-1]$ or by 0 if the corresponding group is missing (cf. [PS08], Chapter 11).

6.2. **Propellers.** Next we will give a combinatorial description of the SI groups and the differential in terms of *propellers* - the "local tropical cycles" in X.

Some more notations:

- Recall that $\Delta, \Delta', \Delta''$ always denote finite faces of X of sedentarity 0.
- We write $\Delta \prec_k \Delta'$ or $\Delta' \succ_k \Delta$, if Δ is a face of Δ' of relative dimension k.
- $\mathrm{Link}_l(\Delta)$ consists of sets $\bar{q} = \{q_1, \ldots, q_l\}$ where each q_i is either a vertex or a divisorial vector, such that $\{\Delta\bar{q}\}$ span a face (infinite in case $\bar{q}$ contains divisorial vectors) adjacent to Δ of relative dimension l.
- $\mathrm{Link}^0_l(\Delta) \subset \mathrm{Link}_l(\Delta)$ consists of those sets $\bar{q} = \{q_1, \ldots, q_l\}$ where q_i are allowed to be only vertices (not the divisorial vectors). In this case $\{\Delta\bar{q}\}$ is finite.
- $\mathrm{Vol}_{\Delta\bar{q}}$ is the integral volume element in the (oriented) face $\{\Delta\bar{q}\}$.
- We will drop the brackets from the notation of the cell $\{\Delta\bar{q}\}$ (e.g., as above) when they become cumbersome.

Let Δ be an oriented finite cell of sedentarity 0. One can naturally identify (see [IKMZ12] for details) the homology groups $H_{2l}(Z_\Delta, \mathbb{Z})$ with the space of local tropical relative l-cycles around Δ. That is, we consider formal $\mathbb{Z}$-linear combinations

$$\sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}$$

of (possibly infinite) cells $\{\Delta\bar{q}\} \succ_l \Delta$ which are balanced along Δ. We call these local cycles *propellers* and abusing the notation we continue denoting this group by $H_{2l}(\Delta)$ (there is no Tate twist however).

Then one can identify the Gysin map $d'' : H_{2i}(\Delta) \to H_{2i-2}(\Delta')$ with the restriction of the propeller to a consistently oriented finite simplex $\Delta' \succ_1 \Delta$. Put together

$$d''\Big(\sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}\Big) = \sum_{q\in \mathrm{Link}^0_1(\Delta)} \Big(\sum_{\bar{r}\in \mathrm{Link}_{l-1}(\Delta q)} \rho_{q\bar{r}}\{\Delta q\bar{r}\}\Big). \tag{6.1}$$

The inclusion map $d' : H_{2l}(\Delta) \to H_{2l}(\Delta')$, where $\Delta' = \Delta \smallsetminus v$ is consistently oriented facet of Δ, is somewhat more tricky. Let $c = \sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta\bar{q}\}$ be an element in $H_{2l}(\Delta)$. For any $\bar{q} \in \mathrm{Link}_l(\Delta)$ let $\{\Delta'\bar{q}\} = \{\Delta\bar{q}\smallsetminus v\}$ be the corresponding cell containing Δ'. Then the image of $d'c$ in $H_{2l}(\Delta')$ will be

$$\sum_{\bar{q}\in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}}\{\Delta'\bar{q}\} + \sum_{\bar{r}\in \mathrm{Link}_{l-1}(\Delta)} \rho_{v\bar{r}}\{\Delta\bar{r}\}, \tag{6.2}$$

where the coefficients $\rho_{v\bar{r}} \in \mathbb{Z}$ are chosen to make the result balanced along Δ'. There is always a unique such choice (cf. [IKMZ12]), namely, the $\rho_{v\bar{r}}$ can be read off from the balancing condition for c along $\{\Delta\bar{r}\}$:

$$\sum_q \rho_{q\bar{r}}\overrightarrow{(\Delta' q)} + \rho_{v\bar{r}}\overrightarrow{(\Delta' v)} = 0 \qquad \mod \{\Delta'\bar{r}\}, \tag{6.3}$$

where $\overrightarrow{(\Delta' q)}$ means the divisorial vector q, or the vector from any vertex of Δ' to q (well defined mod Δ') if q is a vertex, and same for $\overrightarrow{(\Delta' v)}$.

From now on we will not distinguish between the classical geometric Steenbrink-Illusie E_1 complex and its interpretation via complex of propellers. One of the main results in [IKMZ12] is the following statement.

Theorem 6.2 ([IKMZ12]). *$\tilde{E}_2^{q,p} \cong H_q(X; \mathcal{F}_p)$.*

6.3. **Konstruktor.** Now we provide another realization of the Steenbrink-Illusie's E_1 complex in terms of specific tropical simplicial chains. The collection of these chains which we call *konstruktor* forms a subcomplex of $C_\bullet^{bar}(X, \mathcal{F}_\bullet)$, and we can refer to Theorem 6.2 to see that the inclusion is a quasi-isomorphism. A wonderful feature of the konstruktor is that the eigenwave acts on its elements precisely as the monodromy operator ν acts on the terms in the Steenbrink-Illusie's E_1.

Let us fix the first baricentric subdivision of X. We elaborate a little bit on already used notation of the dual cell.

- For a pair $\Delta \succ \Delta'$ of finite simplices of sedentarity 0 in X, and $\bar{q} \in \mathrm{Link}_l(\Delta)$ we let $\hat{\Delta}'_{\Delta\bar{q}}$ denote the dual cell to Δ' in the face $\{\Delta\bar{q}\}$ of X, that is the union of all simplices in the baricentric subdivision containing baricenters of both Δ' and $\{\Delta\bar{q}\}$.
- In the summation formulae to follow we assume the terms with $\hat{\Delta}'_{\Delta\bar{q}}$ are not present if Δ' is not a zero sedentarity finite face of $\{\Delta\bar{q}\}$.

Let Δ be a finite k-simplex of sedentarity 0 in X, and $r \leq k$ a non-negative integer. To any propeller, that is a local tropical l-cycle

$$c = \sum_{\bar{q} \in \mathrm{Link}_l(\Delta)} \rho_{\bar{q}} \{\Delta\bar{q}\} \in H_{2l}(\Delta)$$

we associate a simplicial chain $c[-r] \in C^{bar}_{k+l-r}(X, \mathcal{F}_{l+r})$ as follows (note that $c[0]$ now has other meaning than just c):

$$c[-r] = \sum_{\bar{q} \in \mathrm{Link}_l(\Delta)} \sum_{\substack{\Delta' \prec \Delta \\ \dim \Delta' = r}} (\rho_{\bar{q}} \mathrm{Vol}_{\Delta'\bar{q}}) \hat{\Delta}'_{\Delta\bar{q}}.$$

The orientation of $\hat{\Delta}'_{\Delta\bar{q}}$ is consistent with the original orientation of Δ and the choice of the volume element $\mathrm{Vol}_{\Delta'\bar{q}}$. Clearly for each r between 0 and k the map

$$(\cdot)[-r] : H_{2l}(k) \to C^{bar}_{k+l-r}(X, \mathcal{F}_{l+r})$$

is an injective group homomorphism. We denote its image in $C^{bar}_{k+l-r}(X, \mathcal{F}_{l+r})$ by $K_l(k)[-r]$.

Definition 6.3. The *konstruktor* is the free abelian subgroup in $C_\bullet^{bar}(X, \mathcal{F}_\bullet)$ generated by the $K_l(k)[-r]$ for all k, l and r . Note that $K_l(k)[-r]$ intersect trivially for different triples k, l, r.

Next we want to show that for each p the $\oplus_r K_{p-r}(\bullet - p + 2r)[-r]$ is indeed a subcomplex of $C^{bar}_\bullet(X, \mathcal{F}_p)$ isomorphic to the SI complex $\tilde{E}_1^{\bullet,p}$. This follows at once from comparing the SI differentials $d = d' + d''$ with the simplicial boundary ∂.

Proposition 6.4. $\partial(c[-r]) = (d'c)[-r] + (d''c)[-r-1]$.

Proof. For the proof we need two linear algebra identities. Let σ', σ'' be two opposite faces in a unimodular simplex $\sigma = \{\sigma'\sigma''\}$. Then one has

$$\sum_{\tau'' \prec_1 \sigma''} \mathrm{Vol}_{\sigma'\tau''} = \mathrm{Vol}_{\sigma'} \wedge \mathrm{Vol}_{\sigma''} = \sum_{\tau' \prec_1 \sigma'} \mathrm{Vol}_{\tau'\sigma''},$$

where, say, the left equality easily follows from the case when σ' is a vertex. Here all τ' are oriented consistently with σ', and all τ'' with σ''. We will need this identity in the form

$$\sum_{\Delta' \prec_1 \Delta} \mathrm{Vol}_{\Delta'\bar{q}} = \sum_{q \in \mathrm{Link}_1^0(\Delta)} \mathrm{Vol}_{\Delta\bar{q}\smallsetminus q}, \tag{6.4}$$

where Δ is a finite simplex and $\bar{q} \in \mathrm{Link}_l(\Delta)$. Note that the divisorial vectors (if any) in $\bar{q}$ just multiply both sides of the identity for finite simplices.

The second identity involves a relation among the balancing coefficients $\rho_{v\bar{r}}$ from (6.2) for $c = \sum \rho_{\bar{q}}\{\Delta\bar{q}\}$. One can show (cf. [IKMZ12]) that they satisfy a refined version of (6.3). Namely, for $\Delta' \prec \Delta \prec \{\Delta\bar{q}\}$ we have

$$\sum_q \rho_{q\bar{r}} \overrightarrow{(\Delta' q)} + \sum_{v \in \Delta \smallsetminus \Delta'} \rho_{v\bar{r}} \overrightarrow{(\Delta' v)} = 0 \qquad \mod \{\Delta'\bar{r}\}$$

for faces $\Delta' \prec \Delta$ of codimension possibly higher than 1. Multiplying the above by $\mathrm{Vol}_{\Delta'\bar{r}}$ we arrive at

$$\sum_{q \in \mathrm{Link}_1(\Delta)} \rho_{q\bar{r}} \mathrm{Vol}_{\Delta' q\bar{r}} = -\sum_{v \in \Delta \smallsetminus \Delta'} \rho_{v\bar{r}} \mathrm{Vol}_{\Delta' v\bar{r}} . \tag{6.5}$$

Now we are ready to proof the proposition. Let $c = \sum \rho_{\bar{q}}\{\Delta\bar{q}\}$, then we can write

$$c[-r] = \sum_{\substack{\Delta' \prec_{k-r} \Delta \\ \bar{q} \in \mathrm{Link}_l(\Delta)}} (\rho_{\bar{q}} \mathrm{Vol}_{\Delta'\bar{q}}) \hat{\Delta}'_{\Delta\bar{q}}.$$

The topological boundary of each cell $\Delta'_{\Delta\bar{q}}$ consists of two types:

- Type 1: cells in the form $\Delta''_{\Delta\bar{q}}$ for faces $\Delta'' \succ_1 \Delta'$ of $\{\Delta\bar{q}\}$. If the cell $\Delta'_{\Delta\bar{q}}$ includes divisorial directions then its coefficient $\mathrm{Vol}_{\Delta'\bar{q}}$ in $c[-r]$ is divisible by all divisorial vectors. Hence the type 1 part of the boundary $\partial(c[-r])$ is, in fact, supported on the faces $\Delta''_{\Delta\bar{q}}$ for finite Δ''. Thus $\Delta''_{\Delta\bar{q}}$ in the formulae below make sense.
- Type 2: cells in the form $\Delta'_{\Delta\bar{q}\smallsetminus v}$ where v is a vertex or a divisorial vector in $\{\Delta\bar{q}\}$ which is not in Δ'.

Next we show that these two boundary types endowed with the framing correspond to the d'' and d' differentials in the SI complex, respectively, see Figure 6.

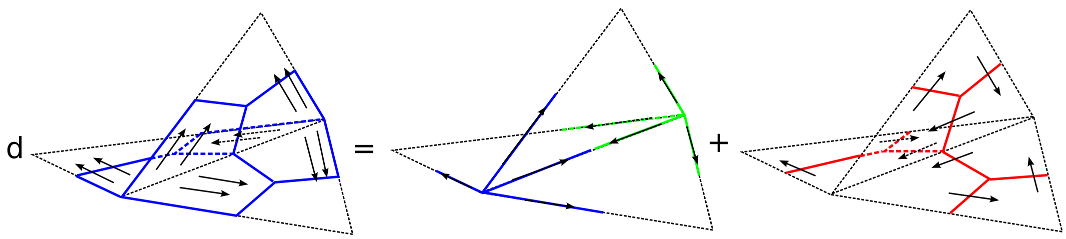


FIGURE 6. $d = d' + d'' : H_2(1) \to H_2(0) \oplus H_0(2)[-1]$. (Framing coefficient vectors are not to scale).

Boundary of type 1:

$$\sum_{\Delta',\bar q}\sum_{q\in\bar q}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\{\hat{\Delta' q}\}_{\Delta\bar q} + \sum_{\bar q}\sum_{\Delta'\prec_1\Delta''\prec\Delta}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\hat\Delta''_{\Delta\bar q}$$

$$= \sum_{q\in \operatorname{Link}^0_1(\Delta)}\Big(\sum_{\substack{\Delta''\prec\Delta q,\ \Delta''\not\prec\Delta\\ \bar r\in\operatorname{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\hat\Delta''_{\Delta q\bar r} + \sum_{\substack{\Delta''\prec\Delta\\ \bar r\in\operatorname{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\Big)\hat\Delta''_{\Delta q\bar r}$$

$$= \sum_{q\in \operatorname{Link}^0_1(\Delta)}\sum_{\substack{\Delta''\prec\Delta q\\ \bar r\in\operatorname{Link}_{l-1}(\Delta q)}}(\rho_{q\bar r}\operatorname{Vol}_{\Delta''\bar r})\hat\Delta''_{\Delta q\bar r}.$$

Here in the second summand we used the identity 6.4 for the pair $\Delta'\prec\Delta''\bar q$. From (6.1) one can easily see that this coincides with $(d''c)[-r-1]$.

Boundary of type 2:

$$\sum_{q\in\operatorname{Link}_1(\Delta)}\sum_{\substack{\Delta'\prec\Delta\\ \bar r\in\operatorname{Link}_{l-1}(\Delta)}}(\rho_{q\bar r}\operatorname{Vol}_{\tau q\bar r})\hat\Delta'_{\Delta\bar r} + \sum_{v\in\Delta}\sum_{\substack{\Delta'\prec\Delta\setminus v\\ \bar q\in\operatorname{Link}_l(\Delta)}}(\rho_{\bar q}\operatorname{Vol}_{\Delta'\bar q})\hat\Delta'_{\Delta\bar q\setminus v}$$

$$= \sum_{\substack{v\in\Delta\\ \Delta'\prec\Delta\setminus v}}\Big(\sum_{\bar r\in\operatorname{Link}_{l-1}(\Delta)}(\rho_{v\bar r}\operatorname{Vol}_{\tau v\bar r})\hat\Delta'_{\Delta\bar r} + \sum_{\bar q\in\operatorname{Link}_l(\Delta)}(\rho_{\bar q}\operatorname{Vol}_{\tau\bar q})\hat\Delta'_{\Delta\bar q\setminus v}\Big).$$

Here in the first summand we used the identity (6.5) for each $\Delta',\bar r$ with the sign compensated by the orientation of $\hat\Delta'_{\Delta\bar r}$ and the choice of $\operatorname{Vol}_{\tau v\bar r}$. Taking the sum of (6.2) over all vertices $v\in\Delta$ we easily identify the last expression with $(d'c)[-r]$. $\square$

Combining the above proposition with Theorem 6.2 we can conclude that the konstruktor complex can be used to calculate the tropical homology groups $H_q(X,\mathcal{F}_p)$:

Corollary 6.5. *The inclusion of the konstruktor $\oplus_r K_{p-r}(\bullet - p + 2r)[-r]$ into $C^{bar}_\bullet(X,\mathcal{F}_p)$ is a quasi-isomorphism for each p.*

Finally, since all infinite cells in the konstruktor chains have coefficients divisible by the divisorial directions we can use the explicit description (4.3) of the eigenwave action on it. Then unveiling the konstruktor definition we arrive at the following.

Proposition 6.6. *For any* $c \in H_{2l}(\Delta)$ *one has* $\phi \cap (c[-r]) = c[-r-1]$.

Now we can combine all above observations to prove the claimed isomorphism

$$\phi^{q-p} : H_q(X; \mathcal{F}_p) \to H_p(X; \mathcal{F}_q).$$

Proof of Theorem 6.1. The cap product action of the eigenwave ϕ^{q-p} on the homology $H_q(X, \mathcal{F}_p)$ can be induced from its action on the konstruktor, which is a simplicial chain subcomplex. But it agrees there with the classical action of the monodromy ν^{q-p} on the E_1 term of the SI spectral sequence. On the other hand it is well known that the ν^{q-p} induces an isomorphism on the associated graded pieces with respect to the monodromy weight filtration on $H_{p+q}(X_t)$, which are calculated on the E_2 term of the SI spectral sequence. ☐

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Université de Genève, Mathématiques, Villa Battelle, 1227 Carouge, Suisse
E-mail address: grigory.mikhalkin@unige.ch

Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506 USA
E-mail address: zharkov@math.ksu.edu